...ITY
...HIGAN
MAR 10 1961
MATHEMATICS
LIB...

# Mathematical Reviews

*Published monthly by* The American Mathematical Society

TABLE OF CONTENTS


| | |
|---|---|
| General | 107 |
| History and Biography | 107 |
| Logic and Foundations | 107 |
| Set Theory | 109 |
| Combinatorial Analysis | 109 |
| Order, Lattices | 111 |
| Classical Algebra | 112 |
| Theory of Numbers | 112 |
| Fields | 115 |
| Linear Algebra | 115 |
| Associative Rings and Algebras | 117 |
| Non-Associative Rings and Algebras | 117 |
| Homological Algebra | 118 |
| Groups and Generalizations | 118 |
| Topological Groups and Lie Theory | 120 |
| Miscellaneous Topological Algebra | 121 |
| Functions of Real Variables | 122 |
| Measure and Integration | 123 |
| Functions of Complex Variables | 125 |
| Special Functions | 129 |
| Ordinary Differential Equations | 132 |
| Partial Differential Equations | 137 |
| Potential Theory | 143 |
| Finite Differences and Functional Equations | 145 |
| Sequences, Series, Summability | 146 |
| Approximations and Expansions | 147 |
| Fourier Analysis | 149 |
| Integral Transforms and Operational Calculus | 151 |
| Integral and Integrodifferential Equations | 153 |
| Functional Analysis | 154 |
| Calculus of Variations | 159 |
| Geometry | 159 |
| Convex Sets and Geometric Inequalities | 162 |
| Differential Geometry | 162 |
| General Topology, Point Set Theory | 167 |
| Algebraic Topology | 169 |
| Differential Topology | 171 |
| Author Index | 177 |


Volume 22, Number 2A February 1961 Reviews 659–993

# MATHEMATICAL REVIEWS

*Published by*

THE AMERICAN MATHEMATICAL SOCIETY, 190 Hope St., Providence 6, R.I.

*Sponsored by*

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
THE INSTITUTE OF MATHEMATICAL STATISTICS
THE EDINBURGH MATHEMATICAL SOCIETY
SOCIÉTÉ MATHÉMATIQUE DE FRANCE
DANSK MATEMATISK FORENING
HET WISKUNDIG GENOOTSCHAP TE AMSTERDAM
THE LONDON MATHEMATICAL SOCIETY
POLSKIE TOWARZYSTWO MATEMATYCZNE
UNIÓN MATEMÁTICA ARGENTINA
INDIAN MATHEMATICAL SOCIETY
UNIONE MATEMATICA ITALIANA
THE SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

*Edited by*

E. Hille W. S. Massey J. V. Wehausen

S. H. Gould, *Executive Editor*

I. Barsotti, Chandler Davis, W. Freiberger and J. A. Zilber, *Associate Editors*

H. A. Pogorzelski, *Copy Editor*

*Editorial Office*

MATHEMATICAL REVIEWS, 190 Hope St., Providence 6, R.I.

*Subscription:* Price $50 per year ($25 per year to individual members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to the American Mathematical Society, 190 Hope St., Providence 6, R.I.

The preparation of the reviews appearing in this publication is made possible by support provided by a grant from the National Science Foundation. The publication was initiated with funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers or proprietors of the publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.


---

Mathematical Reviews is published in 1961 in twelve monthly issues, each in two parts, A and B; and a single index issue covering both parts. Reviews and pages are numbered consecutively with respect to the issue order 1A, 1B, . . . , 12A, 12B. When the letter A or B is prefixed to a review number, it indicates in which part the review appears.

Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci.(7) **4 (82)** (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this is a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place. The symbol ★ precedes the title of a book or other non-periodical which is being reviewed as a whole.

References to reviews in Mathematical Reviews before volume 20 (1959) are by volume and page number, as MR **19**, 532: from volume 20 on, by volume and review number, as MR **20** #4387. Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.


*Copyright © 1961 by the American Mathematical Society*



*Printed in Great Britain by William Clowes and Sons, Limited, London and Beccles*

# Mathematical Reviews

Vol. 22, No. 2A February, 1961 Reviews 659–993

## GENERAL

See also 989.

659:
**Littlewood, D. E. ★The skeleton key of mathematics: A simple account of complex algebraic theories.** Harper Torchbooks/The Science Library. Harper & Brothers, New York, 1960. 138 pp. $1.25.
Reprinting. Previously published by Hutchinson's University Library, London, 1949 [MR 11, 710].

660:
**Monjallon, Albert. ★Introduction aux mathématiques modernes.** Librairie Vuibert, Paris, 1960. 180 pp. 2000 francs.
Elementary textbook on basic concepts: sets, relations, functions, logical calculus, and axiomatics, the latter illustrated for a commutative group. The treatment is informal (e.g., the integers are taken for granted) and detailed, with a great many diagrams, examples and simple exercises.

661:
**Landau, Edmund. ★Grundlagen der Analysis (das Rechnen mit ganzen, rationalen, irrationalen, komplexen Zahlen).** 3rd ed. Chelsea Publishing Co., New York, 1960. 173 pp. Paperbound: $1.95.
Includes a complete German–English vocabulary of the text, directed to the novice in mathematical German, as well as an English translation (together with the original German) of the prefaces. An English translation of the whole [Chelsea Publishing Co., New York, 1951] is listed in MR 12, 397.

## HISTORY AND BIOGRAPHY

662:
**Miller, Maximilian. Leibniz' erster Entwurf zur Begründung der Infinitesimalrechnung.** Wiss. Z. Hochsch. Verkehrswes. Dresden 5 (1957), no. 2, 285–291.
The author gives a translation of a letter written originally by Leibniz in French, in which a method of quadrature for quite general curves—and therewith the beginnings of calculus—are presented. The article has numerous explanatory footnotes, presenting and proving statements of the letter in modern notation.
*L. C. Maximon* (Washington, D.C.)

663:
**Grünbaum, Adolf. Conventionalism in geometry.** The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 204–222. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.
This penetrating research into Poincaré's conventionalism suffers from the circumstance that Poincaré's views dating from the nineties of the past century and published in several periodicals are judged on the evidence of a 1946 third hand English adaptation of the original work. The author's view on Poincaré is anachronistic, as though Poincaré wrote after Hilbert's *Grundlagen der Geometrie* [8th ed., Teubner, Stuttgart, 1956; MR 18, 227], instead of before. So the author describes Poincaré as a supporter of Riemann against Helmholtz, whereas the real situation was just the converse.—The author gives a circumstantial refutation of what he calls Carnap's and Reichenbach's assertion that vanishing of the curvature tensor implies constancy of the $g_{ik}$. The reviewer could not find any indication of such a trivially false assertion in Carnap's and Reichenbach's work. Maybe the author is misled by the ordinary terminology which uses the term Riemannian geometry also for a class of isometric Riemannian geometrics.
*H. Freudenthal* (Utrecht)

664:
**Cooke, R. G. Obituary: Paul Dienes.** J. London Math. Soc. 35 (1960), 251–256.

665:
**Saxer, Walter. Prof. Dr. Louis Kollros: 1878–1959.** Actes Soc. Helv. Sci. Nat. 139 (1959), 423–426. (1 plate)

666:
**Ursell, F.; Olver, F. W. J. Obituary: Roger Chapman Thorne.** J. Austral. Math. Soc. 1 (1959/61), 255–256.

## LOGIC AND FOUNDATIONS

667:
**Angell, R. Bradshaw. Note on a less restricted type of rule of inference.** Mind 69 (1960), 253–255.
It is noted that modus ponens may be replaced in a

propositional logic by the rule *124 of *Mathematical Logic* [W. V. Quine, Harvard Univ. Press, Cambridge, Mass., 1951; MR **13**, 613] and that similarly a substitution of $S'$ for $(S \supset S')$ in a formula when $S$ is a tautology does not alter the truth table for the formula.

*P. C. Gilmore* (Yorktown Heights, N.Y.)

668:

**Dunham, B.; Fridshal, R. The problem of simplifying logical expressions.** J. Symb. Logic **24** (1959), 17–19.

Die Quineschen Methoden zur Vereinfachung logischer Normalformen [Quine, Amer. Math. Monthly **62** (1955), 627–631; MR **17**, 814] benutzen eine systematische Prüfung aller Disjunktionen von Primimplikanten. Verff. zeigen, wie man leicht zu Formeln kommen kann, bei denen die Methode praktisch undurchführbar wird. Schon bei 6 Variablen können es bis zu 90 Primimplikanten sein, d.h. es wären $2^{90}$ Disjunktionen zu prüfen. Mit jeder weiteren Variablen wird die Maximalzahl der Primimplikanten mehr als verdoppelt. Verff. empfehlen daher die Entwicklung von Approximationsmethoden.

*P. Lorenzen* (Kiel)

669:

**Hlodovskiĭ, I. N. A new proof of the consistency of arithmetic.** Uspehi Mat. Nauk **14** (1959), no. 6 (90), 105–140. (Russian)

This is a posthumously published paper (written in 1946–48), with a preface, notes, and corrections by A. S. Esenin-Vol'pin. The methods used are similar to those found in the consistency proof of K. Schütte [Math. Ann. **122** (1951), 369–389; MR **13**, 615].

Let $H$ be a first-order system for arithmetic (Peano's postulates), with definitions permitted of arbitrary recursive functions (by means of their defining systems of equations), and also including predicate variables with and without arguments. For each formula $\mathfrak{A}$ of $H$, define the reduct $R(\mathfrak{A})$ as follows: If $X$ is an atomic formula, $R(X)$ is $X$ and $R(\overline{X})$ is $\overline{X}$. Also, $R(\mathfrak{A} \vee \mathfrak{B})$ is $R(\mathfrak{A}) \vee R(\mathfrak{B})$; $R(\mathfrak{A} \,\&\, \mathfrak{B})$ is $R(\mathfrak{A}) \,\&\, R(\mathfrak{B})$; $R(\overline{\overline{\mathfrak{A}}})$ is $R(\mathfrak{A})$; $R(\overline{\mathfrak{A} \vee \mathfrak{B}})$ is $R(\overline{\mathfrak{A}}) \,\&\, R(\overline{\mathfrak{B}})$; $R(\overline{\mathfrak{A} \,\&\, \mathfrak{B}})$ is $R(\overline{\mathfrak{A}}) \vee R(\overline{\mathfrak{B}})$; $R(\mathfrak{A} \to \mathfrak{B})$ is $R(\overline{\mathfrak{A}}) \vee R(\mathfrak{B})$; $R((x)\mathfrak{A}(x))$ is $(x)R(\mathfrak{A}(x))$; $R((Ex)\mathfrak{A}(x))$ is $(Ex)R(\mathfrak{A}(x))$; $R(\overline{(x)\mathfrak{A}(x)})$ is $(Ex)R(\overline{\mathfrak{A}(x)})$; and $R(\overline{(Ex)\mathfrak{A}(x)})$ is $(x)R(\overline{\mathfrak{A}(x)})$.

Let $N$ be a formal system which has the same symbols as $H$ except that the logical connectives $\to$ and $\overline{\phantom{x}}$ are omitted, and, for each primitive predicate $A$, a new predicate symbol $\overline{A}$ is introduced. (The primitive predicates are "$=$" and the predicate variables.) For each formula $\mathfrak{A}$ of $N$, define the conjugate $\mathfrak{A}^*$ of $\mathfrak{A}$ as follows: $A^*$ is $\overline{A}$ and $(\overline{A})^*$ is $A$; $(\mathfrak{A} \vee \mathfrak{B})^*$ is $\mathfrak{A}^* \,\&\, \mathfrak{B}^*$; $(\mathfrak{A} \,\&\, \mathfrak{B})^*$ is $\mathfrak{A}^* \vee \mathfrak{B}^*$; $((x)\mathfrak{A}(x))^*$ is $(Ex)\mathfrak{A}^*(x)$; and $((Ex)\mathfrak{A}(x))^*$ is $(x)\mathfrak{A}^*(x)$. The axioms of $N$ are: $\mathfrak{A} \vee \mathfrak{A}^*$; $a = a$; $a \neq b \vee \mathfrak{A}^*(a) \vee \mathfrak{A}(b)$; $0 \neq 0'$; and defining equations for arbitrary recursive functions. ("$a$" and "$b$" are free variables.) The rules of inference are:

$$\text{II}_1\colon \frac{(\mathfrak{A} \vee \mathfrak{B}) \vee \theta}{(\mathfrak{B} \vee \mathfrak{A}) \vee \theta}; \quad \text{II}_2\colon \frac{(\mathfrak{A} \vee \mathfrak{B}) \vee \mathfrak{C}}{\mathfrak{A} \vee (\mathfrak{B} \vee \mathfrak{C})}; \quad \text{II}_3\colon \frac{\mathfrak{A}}{\mathfrak{A} \vee \mathfrak{B}};$$

$$\text{II}_4\colon \frac{(\theta \vee \mathfrak{B}) \vee \mathfrak{B}}{\theta \vee \mathfrak{B}}; \quad \text{III}\colon \frac{\mathfrak{A} \vee \theta;\ \mathfrak{B} \vee \theta}{(\mathfrak{A} \,\&\, \mathfrak{B}) \vee \theta}; \quad \text{IV}_1\colon \frac{\mathfrak{A}(a) \vee \theta}{(x)\mathfrak{A}(x) \vee \theta};$$

$$\text{IV}_2\colon \frac{\mathfrak{A}(\tau) \vee \theta}{(Ex)\mathfrak{A}(x) \vee \theta}; \quad \text{V (Induction)}\colon \frac{\mathfrak{A}(0);\ \mathfrak{A}^*(a) \vee \mathfrak{A}(a')}{\mathfrak{A}(a)};$$

$$\text{VI (cut)}\colon \frac{\mathfrak{A} \vee \theta;\ \mathfrak{A}^* \vee \Delta}{\theta \vee \Delta}.$$

In $\text{IV}_1$, "$a$" is not free in $\theta$. In $\text{IV}_2$, "$\tau$" is any term. It is easy to show that if $\mathfrak{A}$ is provable in $H$, then $R(\mathfrak{A})$ is provable in $N$. Hence, if $N$ is consistent (in the sense that not all formulas are theorems), then $H$ is consistent.

Let $C$ be a system having the same symbols as $N$ and the following axioms and rules. Axioms: If $A$ is a predicate variable without arguments, $A \vee \overline{A}$; if $\tau$ and $\nu$ are terms, $\tau = \nu \vee \tau \neq \nu$; $\tau = \tau$; $0 \neq 0'$; $(\tau \neq \nu \vee \overline{A}(\tau)) \vee A(\nu)$, where $A(x)$ is a predicate variable with one or more arguments; defining equations for arbitrary recursive functions. Rules: $\text{II}_1$, $\text{II}_2$, $\text{II}_3$, $\text{II}_4$, $\text{III}_1$, and $\text{IV}_1$ of $N$, plus:

$$\text{IV}_2'\colon \frac{\mathfrak{A}(z) \vee \theta}{(Ex)\mathfrak{A}(x) \vee \theta}$$

where $z$ is a free variable or numeral;

$$\text{V}'\colon \frac{\tau \neq \nu \vee \theta}{(\rho \neq \tau \vee \rho \neq \nu) \vee \theta}; \quad \text{VI}'\colon \frac{(\tau = \nu) \vee \theta;\ (\tau \neq \nu) \vee \Delta}{\theta \vee \Delta};$$

$$\text{VII (Carnap's Rule)}\colon \frac{\mathfrak{A}(0),\ \mathfrak{A}(1),\ \mathfrak{A}(2), \cdots, \mathfrak{A}(n), \cdots}{\mathfrak{A}(a)}.$$

(In V'–VI', $\rho$, $\tau$, $\nu$ are terms.) Given a proof in $C$, we assign ordinals to each inference in the proof. An axiom is assigned 0. A single premiss inference $\frac{\mathfrak{B}}{\mathfrak{A}}$ is assigned $\alpha+1$ if $\alpha$ is the ordinal of the inference yielding $\mathfrak{B}$. A two-premiss inference $\frac{\mathfrak{B};\ \mathfrak{C}}{\mathfrak{A}}$ is assigned the natural sum $\alpha+\beta+1$ if $\alpha$ and $\beta$ are the ordinals of the inferences yielding $\mathfrak{B}$ and $\mathfrak{C}$ respectively. The ordinal of Carnap's Rule VII $\frac{\mathfrak{A}(0),\ \mathfrak{A}(1), \cdots}{\mathfrak{A}(a)}$, where the ordinal of the inference yielding $\mathfrak{A}(n)$ is $\alpha_n$, is the least ordinal greater than all $\alpha_n$. This assignment of ordinals makes possible the use of transfinite induction in proving many of the results mentioned below. (Because of Carnap's Rule, the notion of proof in $C$ is unclear. The author seems to assume that we can ascribe ordinals in the above manner to any given proof. For a somewhat clearer exposition, see Schütte's paper cited above.)

Let $\mathfrak{A}$ be called a disjunctive component (d.c.) of $\mathfrak{A}$; let $\mathfrak{A}$ and $\mathfrak{B}$ be d.c.'s of $\mathfrak{A} \vee \mathfrak{B}$; if $\mathfrak{A}$ is a d.c. of $\mathfrak{B}$, and $\mathfrak{B}$ is a d.c. of $\mathfrak{C}$, then $\mathfrak{A}$ is a d.c. of $\mathfrak{C}$. Ten pages of the paper are devoted to showing that, in $C$, one can derive a stronger cut rule $\text{VI}_1$. Let $\mathfrak{A}$ be a d.c. of $\mathfrak{R}$, and let $\mathfrak{A}^*$ be a d.c. of $\mathfrak{R}^*$. Let $\tilde{\mathfrak{R}}_{\mathfrak{A}}$ be the result of erasing some occurrences of $\mathfrak{A}$ as a d.c. in $\mathfrak{R}$; and let $\tilde{\mathfrak{R}}_{\mathfrak{A}^*}$ be defined similarly. Then, rule $\text{VI}_1$ is $\frac{\mathfrak{R};\ \mathfrak{R}^*}{\tilde{\mathfrak{R}}_{\mathfrak{A}} \vee \tilde{\mathfrak{R}}_{\mathfrak{A}^*}}$. Using this result, the author proves that every theorem of $N$ is a theorem of $C$. Hence, if $C$ is consistent, so is $N$.

If $\tau$ and $\nu$ are terms without free variables, then $\tau = \nu$ and $\tau \neq \nu$ are called elementary arithmetic (e.a.) formulas. Also, if $\mathfrak{A}$ and $\mathfrak{B}$ are e.a., so is $\mathfrak{A} \vee \mathfrak{B}$. It is shown that any provable e.a. formula can be given a proof consisting entirely of e.a. formulas. The notion of "true" e.a. formula is introduced, and it is noted that every provable e.a. formula is true. Hence the false e.a. formula $0 \neq 0$ is unprovable. Consequently, $C$ is consistent, and therefore $N$ and $H$ are also.

The system $C$ is a special case of a class of systems studied by P. S. Novikov [Mat. Sb. (N.S.) **12** (**54**) (1943), 231–261; MR **5**, 197]. In his preface, Esenin-Vol'pin notes that the author's proof shows the $\omega$-consistency of formal arithmetic (since Carnap's Rule is included in $C$), and he states that the proof requires induction on the ordinals of the second number class, but that if constructivist restrictions were placed on Carnap's Rule, then induction only on constructive ordinals would be needed.

*E. Mendelson* (New York)

670:

**Rose, Alan. Sur les schémas d'axiomes pour les calculs propositionnels à $m$ valeurs ayant des valeurs surdésignées.** C. R. Acad. Sci. Paris **250** (1960), 790–792.

In an $m$-valued propositional calculus, call the values $1, \cdots, s$ "superdesignated", and the values $s+1, \cdots, t$ "designated" $(1 \leq s < t < m)$. Then a formula is called a tautology if either it takes a superdesignated value at least once or it always takes designated values. The author shows that, given certain general conditions on the primitive functors, an axiom system can be constructed in which the set of theorems is identical with the set of tautologies.

*E. Mendelson* (New York)

671:

**Meschkowski, Herbert. ★Wandlungen des mathematischen Denkens: Eine Einführung in die Grundlagenprobleme der Mathematik.** 2te, durchgesehene u. erweiterte Auflage. Friedr. Vieweg & Sohn, Braunschweig, 1960. vii+141 pp. (4 plates) DM 12.80.

Topics added to the first edition [1956; MR **20** #1] include: Bourbaki (in the chapter on formalism); Lorenzen's operational mathematics; and an annex on "Education in the age of automation".

## SET THEORY

672:

**Bagemihl, Frederick. Concerning the continuum hypothesis and rectilinear sections of spatial sets.** Arch. Math. **10** (1959), 360–362.

Starting from some results of Sierpiński [Fund. Math. **38** (1951), 1–13; MR **14**, 26; theorem 6] the author proves the following. Theorem 1: Let $n$ be a natural number and $S = E_1 \cup E_2 \cup E_3$ a decomposition of the Euclidean space $S$ such that every $x$-line (=line parallel to the $x$-axis) intersects $E_1$ in at most $n$ points and every $y$-line intersects $E_2$ in a nowhere dense set of the line; there are subsets $P$, $P'$, $P''$ of the $xy$-plane, each of which is everywhere of cardinality $\mathfrak{c}$, everywhere of category II, and everywhere of positive exterior measure, such that every $z$-line intersecting $P$ [resp. $P'$, resp. $P''$] intersects $E_3$ in a set of cardinality $\mathfrak{c}$ [category II, positive exterior measure]. $\mathfrak{A}_K$: $S = E_1 \cup E_2 \cup E_3$, and every $x$-line intersects $E_1$ in a finite set, every $y$-line intersects $E_2$ in $\leq \aleph_0$ points, every $z$-line intersects $E_3$ in a set of category I. $\mathfrak{R}^*$: The line is not the union of $<\mathfrak{c}$ sets of category I. The conjunction of $\mathfrak{A}_K$ and $\mathfrak{R}^*$ is equivalent to the continuum hypothesis H (Th. 2). Replacing in $\mathfrak{A}_K$ the condition "category I" by "measure 0" resp. "power $<\mathfrak{c}$" one gets the statements $\mathfrak{A}_M$, $\mathfrak{A}_P$. Then $H \Leftrightarrow \mathfrak{A}_M \,\&\, \mathfrak{R}^*$, where $\mathfrak{R}^*$ means that the line is not the union of $<\mathfrak{c}$ sets of measure 0 (Th. 3). $H \Leftrightarrow \mathfrak{A}_P \,\&\, (\mathfrak{c}$ is regular) (Th. 4).

*Đ. Kurepa* (Zagreb)

## COMBINATORIAL ANALYSIS

See also B1048.

673:

**Goldhaber, J. K. Integral $p$-adic normal matrices satisfying the incidence equation.** Canad. J. Math. **12** (1960), 126–133.

A $v$, $k$, $\lambda$ configuration is an arrangement of $v$ elements into $v$ sets such that every set contains exactly $k$ distinct elements and such that every pair of sets has exactly $\lambda = k(k-1)/(v-1)$ elements in common, where $0 < \lambda < k < v$. Let $B$ be the $v$ by $v$ matrix with $k$ in every position on the main diagonal and $\lambda$ in all other positions. Suppose that there exists a $v$ by $v$ matrix $A$ with rational elements such that $AIA^T = B$. Here $A^T$ denotes the transpose of $A$ and $I$ is the $v$ by $v$ identity matrix. The matrix $I$ is said to represent $B$ rationally. The existence of a $v$, $k$, $\lambda$ configuration implies that $I$ represents $B$ rationally and this in turn yields the Bruck-Chowla-Ryser criterion for the nonexistence of these configurations. The author's central theorem is the following. If $I$ represents $B$ rationally, then for every prime $p$ there exists a matrix $A$ with elements in the ring $R(p)$ of $p$-adic integers such that

$$AA^T = A^TA = B.$$

The proof makes extensive use of the arithmetic theory of quadratic forms and the theorem does not establish the nonexistence of new configurations. But the theorem does imply that if $I$ represents $B$ rationally, then $I$ represents $B$ rationally without essential denominator. Also if $I$ represents $B$ rationally, then there exists a form in the genus of $I$ which represents $B$ integrally.

The author also studies a modified incidence equation and proves the following. Let $t = a/b > 1/v$ be rational and such that $(av-b)b$ is odd. Let $S$ be the $v$ by $v$ matrix of 1's. If $I$ represents $B$ rationally then $I - tS$ represents $B - tk^2S$ integrally. A concluding theorem gives sufficient conditions for the existence of a $v$, $k$, $\lambda$ configuration in terms of certain types of integral solutions of the matric equation

$$X^T(I - tS)X = B - tk^2S.$$

*H. J. Ryser* (Columbus, Ohio)

674:

**Parker, E. T. Construction of some sets of mutually orthogonal latin squares.** Proc. Amer. Math. Soc. **10** (1959), 946–949.

The author proves by direct construction the following two theorems. I. If there exists a balanced incomplete block design with parameters $(v, k, 1)$ and a projective geometry with $k^2+k+1$ points, then there exist $k-2$ mutually orthogonal Latin-squares of order $v$. II. If $m$ is a Mersenne prime or $m+1$ a Fermat prime, then there exists even a set of $m$ mutually orthogonal Latin-squares of side $m^2+m+1$. (This is one more than could be expected from I.)

The notation in II can be so arranged that one of the squares is cyclic with digit one in the principal diagonal. The other squares have the following property: If the cell $(x_1, x_2)$ contains $x$, then the cell $(x_1+1, x_2+1)$ contains $x+1$, where all calculations are mod. $m^2+m+1$. Thus they can easily be completed if their first rows are given. Except for $m=3$ the methods of MacNeish and related methods give only an orthogonal pair under the conditions of II. *H. B. Mann* (Columbus, Ohio)

675:

**Ross, Ian C.; Harary, Frank. A description of strengthening and weakening members of a group.** Sociometry **22** (1959), 139–147.

Let $D$ be a directed graph on the points $S$, and $D^-$ the graph on $S$ having lines $(i, j)$ and $(j, i)$ if and only if $D$ has at least one. Define the Boolean matrix $R[R^-]$ by $R_{ij}=1$ if there is a directed path in $D[D^-]$ from $i$ to $j$ $(i, j \in S)$ and 0 otherwise. Define $D \in U_3$ if $R_{ij}=1$ for all $i, j \in S$; $D \in U_2$ if $D \notin U_3$ and $R_{ij}=1$ or $R_{ji}=1$ for all $i, j \in S$; $D \in U_1$ if $D \notin U_2$ and for every $T \subset S$ there is at least one line between $T$ and $S-T$; $D \in U_0$ if there exists $T \subset S$ such that there is no line between $T$ and $S-T$. A point $i$ is of type $(m, n)$ if $D \in U_m$ and $D-i \in U_n$. All types save (1, 3) can occur. The $U_i$ and some of the possible types of points are characterized in a nearly trivial fashion in terms of $R$ and $R^-$. *R. D. Luce* (Philadelphia, Pa.)

676:

**Cobb, S. M. On powers of matrices with elements in the field of integers modulo 2.** Math. Gaz. **42** (1958), 267–271.

This is an elementary but interesting contribution to the theory of matrices with elements 0 and 1 for which $1+1=0$. It complements but does not overlap recent results in this field obtained by R. D. Luce [Proc. Amer. Math. Soc. **3** (1952), 382–388; MR **14**, 347], D. Rosenblatt [Naval Res. Logist. Quart. **4** (1957) 151–167; MR **19**, 889] and Ross and and Harary [#675 above]. Only square matrices are considered. Lemma: If $A$ is a matrix of order $n$ with elements in any field and $s$ is the smallest integer for which $A^s=0$, then $s \leqq n$. Theorem: For any $n$, there exists a binary matrix $A$ of order $n$ such that the smallest value of $r$ for which $A^r=I$ is $r=2^n-1$. *F. Harary* (Ann Arbor, Mich.)

677:

**Roy, Bernard. Transitivité et connexité.** C. R. Acad. Sci. Paris **249** (1959), 216–218.

Let $D$ be a directed graph (digraph) with $n$ points $x_i$ whose adjacency matrix is $A=(a_{ij})$, i.e., $a_{ij}=1$ if $D$ contains the line from $x_i$ to $x_j$ and $a_{ij}=0$ otherwise. Let $T_iA$ be the matrix obtained from $A$ by reproducing the 1's in the $i$th row on every row having a 1 in the $i$th column. Thus $T_iA$ is the adjacency matrix of the digraph obtained from $D$ by adjoining to $D$ all those directed paths of length 2 through the point $x_i$. Then $T_i$ is idempotent and $T_i$, $T_j$ commute. The main theorem is that the matrix $B=T_nT_{n-1}\cdots T_2T_1A$ gives the transitive closure of $D$, i.e., is the adjacency matrix of the reachability digraph $R(D)$ in which there is a line from $x_i$ to $x_j$ whenever there is a path in $D$ from $x_i$ to $x_j$.

A digraph is strongly connected (or strong) if there is a path from any $x_i$ to any $x_j$. As a corollary, it is stated that $D$ is strong if and only if matrix $B$ consists entirely of 1's. Using the alternative form $B=A^{n-1}$, this simple observation had already been mentioned by Rosenblatt, Ross and Harary [references in #676 above] and Harary [#678 below]. *F. Harary* (Ann Arbor, Mich.)

678:

**Harary, Frank. A graph theoretic method for the complete reduction of a matrix with a view toward finding its eigenvalues.** J. Math. Phys. **38** (1959/60), 104–111.

This paper presents a method for finding the irreducible submatrices of an $n \times n$ matrix $M$, under similarity transformation by permutation matrices. Let $A$ be obtained from $M$ by replacing all non-zero entries of $M$ by 1 and putting 1 in every place on the diagonal. Then $A$ is the adjacency matrix of the directed graph $D(M)$ associated with $M$. Compute powers of $A$ using the boolean addition rule, $1+1=1$. Then $A^m=A^{n-1}=R=(r_{ij})$, for all $m \geqq n-1$, and $r_{ij}=1$ if and only if there is a directed path from the point $a_i$ to the point $a_j$ in $D(M)$. Let $S_1$ be the collection of indices $i$ such that $r_{i1}=r_{1i}=1$. These indices mark the points of $D(M)$ in the same strong component as $a_1$, and the corresponding rows and columns of $M$ form an irreducible submatrix of $M$. Delete the corresponding rows and columns from $R$, and form $S_2$. Continuing in this way, all the strong components of $D(M)$, and the corresponding submatrices $M_1, M_2, \cdots, M_r$ of $M$, are found. The set of eigenvalues of $M$ is the union of the sets of eigenvalues of the submatrices $M_i$, including multiplicity.
*A. M. Duguid* (London)

679:

**Tutte, W. T. On the symmetry of cubic graphs.** Canad. J. Math. **11** (1959), 621–624.

Supplementing an earlier investigation of his concerning $s$-regular cubic graphs [Proc. Cambridge Philos. Soc. **43** (1947), 459–474; MR **9**, 97] the author shows that the class of these graphs is rather larger than one would expect; in fact, that it consists of all cubical graphs whose automorphism groups act transitively on the sets of oriented edges. *G. Sabidussi* (New Orleans, La.)

680:

**Dijkstra, E. W. Some theorems on spanning subtrees of a graph.** Nederl. Akad. Wetensch. Proc. Ser. A **63**= Indag. Math. **22** (1960), 196–199.

The following algorithm is shown to yield the shortest spanning subtree of a graph of which no two edges have the same length: order the edges arbitrarily and lay them down in order until a loop is formed; discard the largest edge of this (unique) loop and continue adding edges in this manner until all edges have either been added or discarded. The undiscarded edges form the desired tree. Using this algorithm and assuming that distinct trees have distinct lengths, it is shown that the second and third shortest trees differ from the shortest in at most one edge and more generally the $k$th shortest tree differs from the shortest in at most $\log_2 k$ edges. Also any spanning tree except the shortest contains at most one edge not belonging to a shorter tree. *D. Gale* (Providence, R.I.)

681:

**Lupanov, O. B. Asymptotic estimates of the number of graphs with $n$ edges.** Dokl. Akad. Nauk SSSR **126** (1959), 498–500. (Russian)

Asymptotische Abschätzungen sind (ohne Beweis) angegeben für die Anzal $G(n)$ von nichtisomorphen Graphen mit $n$ Kanten ohne isolierte Knotenpunkte, sowie für dei Anzahl von solchen Graphen, die noch einige der drei folgenden Eigenschaften besitzen: zusammenhängend, ohne Zweiecke und ohne Schlingen zu sein. Es gilt $G(n)=[2e^{-1}n(\ln n)^{-2}\gamma(n)]^n$ mit

$$2(\ln\ln n)(\ln n)^{-1} \lesssim \gamma(n)-1 \lesssim 4(\ln\ln n)(\ln n)^{-1},$$

wo $\alpha_n \lesssim \beta_n$ die Bedeutung $\limsup \alpha_n(\beta_n)^{-1} \leq 1$ hat. Dieselben Abschätzungen treten auch in den anderen obenerwähnten Fällen ein. Hieraus folgt, dass asymptotisch $[G(n)]^{1/n} \sim 2e^{-1}n(\ln n)^{-2}$ gilt. Fast sämtliche Graphen mit $n$ Kanten haben asymptotisch $2n(\ln n)^{-1}$ Knotenpunkte, wobei der mittlere Grad eines Knotenpunktes asymptotisch gleich $\ln n$ ist. Zwei Hilfssätze, auf welchen der Beweis dieser Sätze beruht, sind angeführt. *M. Fiedler* (Prague)

## ORDER, LATTICES

682:

**Stelleckiĭ, I. V. Nilpotent lattices.** Dokl. Akad. Nauk SSSR **128** (1959), 680–683. (Russian)

A multiplication lattice is defined as a complete lattice $L$ with a binary operation $ab$ such that $a0=0a=0$ for all $a \in L$. An element of $L$ is finitary if, whenever it precedes the join of a directed set, it precedes some member of this set. Consider condition (K): Every element of the complete lattice $L$ is a join of finitary elements [cf. Kerstan, *Bericht über die Mathematiker-Tagung in Berlin*, 1953, pp. 49–57, Deutscher Verlag, Berlin, 1953; MR **16**, 1083]. The author gives some further conditions which hold in the multiplication lattice of subgroups of a group, or of subrings of a ring (with respect to multiplication or commutation). In any multiplication lattice with greatest element $I$, the lower right-central chain is defined inductively by $I^\alpha = I^{\alpha-1}I$ if $\alpha-1$ exists, and $I^\alpha = \bigcap_{\beta<\alpha} I^\beta$ otherwise. The lower left-central chain is defined by symmetry and upper right- and left-central chains are defined by analogy [cf. Birkhoff, *Lattice theory*, revised ed., Amer. Math. Soc., New York, 1948; MR **10**, 673; p. 204f.]. Various concepts of nilpotence can now be defined, such as $ZA$, $N$, $\tilde{N}$ [cf. Kuroš, *Theory of groups*, Chelsea, New York, 1956; MR **18**, 188] and analogues of known theorems are stated for multiplication lattices, e.g.: every right $ZA$-lattice satisfying (K) and certain other conditions is locally nilpotent. It is also stated that in any right $ZA$-lattice, the upper right-central series terminates in $I$; this and other results are combined to give a solution of Birkhoff's problems 89 and 90 [loc. cit.]. *P. M. Cohn* (Manchester)

683:

**Stelleckiĭ, I. V. On complete lattices represented by sets.** Uspehi Mat. Nauk (N.S.) **12** (1957), no. 6 (78), 177–180. (Russian)

The paper deals with complete lattices which may be represented by sets in such a way that the join of each chain corresponds to the set-theoretical sum of elements of its image and the meet of any set $M$ of elements is represented by the set-theoretical product of elements of the image of $M$.

The class of such lattices follows from the author's theorem: A complete lattice $L$ can be represented by sets in the above sense if and only if the following conditions are satisfied: (1) for $x \in L$ and any chain $\{y_\alpha\}$ in $L$ the equality $x \cap \bigcup_\alpha y_\alpha = \bigcup_\alpha (x \cap y_\alpha)$ holds; (2) any element $a \in L$ is a join of elements of $L$ "weakly attainable from below". (An element $b \in L$ is called weakly attainable from below if, for any chain $\{b_\beta\}$ $(\beta \in B)$ in $L$, $b = \bigcup_\beta b_\beta$ implies $b = b_{\beta_0}$, $\beta_0 \in B$.) The author studies certain properties of such a lattice $L$ and proves that each element of $L$ is a meet of irreducible elements. In $L$ the following conditions are equivalent. (1) $L$ is a distributive lattice; (2) $L$ is a well-distributive lattice, i.e., for $x \in L$ and any set $\{x_\alpha\}$ in $L$, $x \cap \bigcup_\alpha x_\alpha = \bigcup_\alpha (x \cap x_\alpha)$; (3) $L$ is a lattice with relative pseudocomplements (i.e., for $a$, $b \in L$ an element $c \in L$—a pseudocomplement of $a$ with respect to $b$—exists such that $a \cap x \leq b \Leftrightarrow x \leq c$ for each $x \in L$).

*V. Vilhelm* (Prague)

684:

**Sikorski, R. Distributivity and representability.** Fund. Math. **48** (1959), 95–103.

A Boolean algebra $B$ is called $\mathfrak{m}$-representable (where $\mathfrak{m}$ is an infinite cardinal number) if there is an $\mathfrak{m}$-isomorphism (that is, preserving joins of subsets of cardinality at most $\mathfrak{m}$ whenever they exist) of $B$ into an $\mathfrak{m}$-homomorph of an $\mathfrak{m}$-field of sets. In this paper, the relation between the $\mathfrak{m}$-representability and the $\mathfrak{m}$-distributivity of a Boolean algebra is clarified. The main result of the paper generalizes the algebraic conditions of E. C. Smith, Jr. [Ann. of Math. (2) **64** (1956), 551–561; MR **19**, 115] and C. C. Chang [Trans. Amer. Math. Soc. **85** (1957), 208–218; MR **19**, 243] and the topological condition of Pierce [Proc. Amer. Math. Soc. **10** (1959), 42–50; MR **21** #5592] for the $\mathfrak{m}$-representability of a Boolean algebra. A number of interesting subsidiary results are also presented. For example, it is shown that an $\mathfrak{m}$-complete Boolean algebra is $\mathfrak{m}$-distributive if and only if every $\mathfrak{m}$-subalgebra generated by at most $\mathfrak{m}$ elements is atomic, and that every weakly $\mathfrak{m}$-distributive Boolean algebra is $\mathfrak{m}$-representable.

*R. S. Pierce* (Seattle, Wash.)

685:

**Jónsson, Bjarni. Arguesian lattices of dimension $n \leq 4$.** Math. Scand. **7** (1959), 133–145.

It is known [Jónsson, Math. Scand. **1** (1953), 193–206; **2** (1954), 295–314; MR **15**, 389; **16**, 787] that every lattice of commuting equivalence relations is Arguesian, satisfying a certain postulate representative of the lattice of subspaces of an Arguesian projective geometry. The present paper proves the converse for lattices of dimension $n \leq 4$, by proving the stronger statement that every Arguesian lattice with $n \leq 4$ is isomorphic to a sublattice of the lattice of all subspaces of an Arguesian projective geometry of dimension $n-1$. On the way to this, the structure of all modular lattices of dimension $n \leq 4$ is analyzed. *P. M. Whitman* (Silver Spring, Md.)

## CLASSICAL ALGEBRA

686:

**Hampl, Miloslav. The solution of biquadratic equations.** Apl. Mat. **4** (1959), 463–465. (Czech. Russian and English summaries)

A new version of Ferrari's literal solution of the biquadratic equation $f(x)=x^4+a_1x^3+\cdots+a_4=0$. Put $x=y+k$, $p=6f'(k)/f''(k)$, $p^2=f(k)$, $z=y+p/y$; this reduces the equation $f(x)=0$ into $z^2+\frac{1}{2}f''(k)z+\frac{1}{3}f'''(k)-2p=0$, where $k$ is defined by the equation $f(k)(f''(k))^2=36(f'(k))^2$ which turns out to be cubic in $k$.

*H. Schwerdtfeger* (Montreal)

## THEORY OF NUMBERS

687:

**Davenport, H. ★The higher arithmetic: An introduction to the theory of numbers.** Harper Torchbooks/The Science Library. Harper & Brothers, New York, 1960. 172 pp. $1.25.

Reprinting. Previously published by Hutchinson's University Library, London, 1952 [MR **14**, 352].

688:

**Mąkowski, A. On the diophantine equation $2^x+11^y=5^z$.** Nordisk Mat. Tidskr. **7** (1959), 81, 96.

The author states: "The purpose of the present note is to prove the following theorem. The only solutions in non-negative integers $x$, $y$, $z$ of the equation $2^x+11^y=5^z$ are given by $x=2$, $y=0$, $z=1$ and $x=y=2$, $z=3$."

*A. Sklar* (Chicago, Ill.)

689:

**Carlitz, L. Congruence properties of Hermite and Laguerre polynomials.** Arch. Math. **10** (1959), 460–465.

Suppose $n$ and $p$ denote non-negative integers, $m$ and $r$ denote positive integers, $r_1=[(r+1)/2]$, $a$ denotes an indeterminate or a number that is integral (mod $m$), $H_n(a)$ denotes the $n$th Hermite polynomial, and $\Lambda_n^{(\lambda)}(a)=n!L_n^{(\lambda)}(a)$ where $L_n^{(\lambda)}(a)$ denotes the $n$th Laguerre polynomial of order $\lambda$. The author obtains the congruence

$$(*)\quad \sum_{s=0}^{r}(-1)^s\binom{r}{s}H_{n+sm}(a)H_{p+(r-s)m}(a)\equiv 0 \pmod{m^{r_1}},$$

which is a generalization of the result for $p=0$ which he had obtained earlier [Math. Z. **59** (1954), 474–483; MR **15**, 604]. A similar congruence holds if $H_n(a)$ is replaced by $\Lambda_n^{(\lambda)}(a)$. The author first proves

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}(2a)^{(r-s)m}H_{n+sm}(a)\equiv 0 \pmod{m^{r_1}},$$

and then uses this to obtain (*). He also obtains the "mixed" congruence

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}H_{n+sm}(-a)\Lambda_{p+(r-s)m}^{(\lambda)}(2a)\equiv 0 \pmod{m^{r_1}}.$$

*R. P. Kelisky* (Yorktown Heights, N.Y.)

690:

**Carlitz, L. A congruence satisfied by the theta-constant $\vartheta_3$.** Proc. Amer. Math. Soc. **10** (1959), 912–916.

Let $\vartheta_3=\sum_{-\infty}^{\infty}q^{n^2}$ and let $p$ denote an arbitrary odd prime. The following congruence appears incidentally in an earlier paper of the author [Math. Z. **64** (1956), 425–434; MR **17**, 1057]:

$$\text{(i)}\qquad \vartheta_3^{2(p-1)}\sum_{r=0}^{m}\binom{m}{r}^2k^{2r}\equiv 1 \pmod{p},$$

where $p=2m+1$ and $k^2$ has its usual significance in the theory of elliptic functions. The congruence (i) is to be interpreted in the following way. Using the familiar identity

$$\text{(ii)}\qquad k^2=16q\prod_{1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^8,$$

(i) can be written entirely in terms of $q$ or entirely in terms of $k^2$. It follows from (ii) that

$$\text{(iii)}\qquad k^2=\sum_{n=1}^{\infty}a_nq^n \quad (a_1=16),$$

where the $a_n$ are rational integers, and that

$$\text{(iv)}\qquad q=\sum_{n=1}^{\infty}b_nk^{2n} \quad (b_1=1/16),$$

where the denominators of the $b_n$ are powers of 2. If we substitute from (iii) into (i) we get a certain set of congruences; if we substitute from (iv) we get another set. By Lemma 1 of the author's earlier paper these congruences are equivalent. The author now shows how the second substitution can be carried out explicitly.

*A. L. Whiteman* (Princeton, N.J.)

691:

**Cohen, Eckford. A class of residue systems (mod $r$) and related arithmetical functions. I. A generalization of Möbius inversion.** Pacific J. Math. **9** (1959), 13–23.

The author considers so-called conjugate pairs $(P, Q)$ of direct factor sets, having the following properties: (1) $P$ and $Q$ are subsets of the set $Z$ of positive integers. (2) If $n_1, n_2\in Z$ and $(n_1, n_2)=1$, then $n_1n_2\in P \rightleftarrows n_1\in P$, $n_2\in P$; and the corresponding condition holds for $Q$. (3) For every $n\in Z$ there is a unique factorization of the form $n=ab$, $a\in P$, $b\in Q$. A $P$-system (mod $r$) consists of those elements of a reduced residue system mod $r$ which belong to $P$, and $\phi_P(r)$ is the number of elements in the $P$-system. Using a generalized Möbius function $\mu_P(r)$ of the class $P$, the author gives inversion formulas of arithmetic functions corresponding to the classical formulas and particularly a formula for $\phi_P(r)$. Also generalized Ramanujan sums of the classes $P$ are considered and represented by sums involving the function $\mu_P(r)$.

*H. Bergström* (Göteborg)

692:

**Cohen, Eckford. A class of residue systems (mod $r$) and related arithmetical functions. II. Higher dimensional analogues.** Pacific J. Math. **9** (1959), 667–679.

The idea of direct factor systems of numbers presented in #691 above is here generalized to direct factor systems of vectors of numbers. A $\zeta$-function associated with a direct factor set is defined and used to describe the asymptotic density of certain vector sets.

*H. Bergström* (Göteborg)

693:

**Touchard, Jacques. Sur quelques séries de Lambert et de Dirichlet.** Canad. J. Math. **12** (1960), 1–19.

Let $D$ be a squarefree positive integer, let $\chi(n)=(D/n)$ be the Jacobi symbol, and let $2\omega$ be its period as a function of $n$. The author considers the expression $g\{\exp(2\pi i n/2\omega)\}$, which is well known in the theory of the class number of binary quadratic forms with a given discriminant, and which can be considered as a generalized Ramanujan sum. He obtains analogies of Ramanujan's formula

$$\sum c_q(n)q^{-s} = n^{1-s}(\zeta(s))^{-1}\sum_{d|n} d^{s-1}.$$

The paper connects this matter with a general (but rather trivial) formula for sums of the form

$$\sum a_k x^k y^k/(1-x^k)(1-y^k).$$

*N. G. de Bruijn* (Amsterdam)

694:

**Carlitz, L. Note on a theorem of S. Uchiyama.** Acta Arith. **5**, 289–292 (1959).

Let $Z(m, n)$ denote the number of systems of complex numbers $(z_1, \cdots, z_n)$ satisfying the system of equations $s_{m+1}=s_{m+2}=\cdots=s_{m+n-1}=0$, where $s_k=\sum_{i=1}^n z_i^k$ and $m$ is an integer $\geq 0$. Two systems $(z_1, \cdots, z_n)$ and $(z_1', \cdots, z_n')$ are equivalent in $Z(m, n)$ if there exists a complex number $\lambda\neq 0$ such that $f(x; z_1, \cdots, z_n)=f(x; \lambda z_1', \cdots, \lambda z_n')$, where $f(x; z_1, \cdots, z_n)=\prod_{j=1}^n (x-z_j)$. Let $B(m, n)$ denote the number of classes of non-trivial sets relative to this equivalence relation. Recently Uchiyama [Acta Arith. **4** (1958), 240–246; MR **20** #860] proved that

$$(1)\qquad \sum_{d|(m,n)} a(d)B\left(\frac{m}{d}, \frac{n}{d}\right) = \frac{(m+n-1)!}{m!n!},$$

where $a(1)=1$ and $a(n)=n^{-1}\prod_{p|n}(1-p)$. The present author proves that (1) implies the explicit result

$$(2)\qquad B(m, n) = \sum_{d|(m,n)} \frac{\phi(d)}{d} C\left(\frac{m}{d}, \frac{n}{d}\right),$$

where $\phi(d)$ is the Euler $\phi$-function and $C(m, n)=(m+n-1)!/m!n!$. He also shows directly that the right member of (2) is integral.

*A. L. Whiteman* (Princeton, N.J.)

695:

**Feit, Walter; Fine, N. J. Pairs of commuting matrices over a finite field.** Duke Math. J. **27** (1960), 91–94.

Let $P(n)=P(n, q)$ denote the number of ordered pairs of (not necessarily distinct) commuting $n$ by $n$ matrices with elements in the finite field GF($q$). It is proved that

$$P(n) = q^{n^2}f(n)\sum_{\pi(n)} \frac{q^{k(\pi)}}{f(b_1)f(b_2)\cdots f(b_n)},$$

where $f(n)=(1-q^{-1})(1-q^{-2})\cdots(1-q^{-n})$, $f(0)=1$; the summation is over all partitions $n=b_1+2b_2+\cdots$; and $k(\pi)=b_1+b_2+\cdots$. Also, it is proved that

$$\sum_{n=0}^{\infty}\frac{P(n)}{q^{n^2}f(n)}x^n = \prod_{i=1}^{\infty}\prod_{j=0}^{\infty}(1-q^{1-j}x^i)^{-1}.$$

*L. Carlitz* (Durham, N.C.)

696:

**Sierpiński, W. Sur les nombres premiers ayant des chiffres initiaux et finals donnés.** Acta Arith. **5**, 265–266 (1959).

In a previous note the author proved the existence of infinitely many primes whose first $n$ digits or last $n$ digits (with the last digit prime to the base) are preassigned [Matematiche, Catania **6** (1951), 135–137; MR **13**, 822]. In the present note he shows that it is also possible to preassign both the first $m$ digits and the last $n$ digits. This results from a combination of the prime number theorem for arithmetic progressions and Dirichlet's theorem.

*D. H. Lehmer* (Berkeley, Calif.)

697:

**Zeckendorf, E. Familles de nombres premiers.** Bull. Soc. Roy. Sci. Liège **29** (1960), 15–27.

With reservation for the primes 2 and 3, the author proposes to partition the prime numbers in the following three families: (1) the prime numbers $P\equiv 1 \pmod 4$, which can be expressed, as well as their products, by a sum $(a^2+b^2)$ of terms relatively prime on the condition that no prime factor $P'\equiv -1 \pmod 4$ is included; (2) the prime numbers $P\equiv 1 \pmod 6$, which can be expressed, as well as their products, by a sum $(3a^2+b^2)$ of terms relatively prime on the condition that no prime factor $P'\equiv -1 \pmod 6$ is included; and (3) the prime numbers $P\equiv \pm 1 \pmod{12}$. Since the case (1) is known, the case (2) is mainly treated, and the case (3) will be discussed later.

*S. Ikehara* (Tokyo)

698:

**Hornfeck, Bernhard. Über natürliche Zahlen, deren Primteiler in mindestens $k$-ter Potenz auftreten.** Math. Ann. **138** (1959), 442–449.

Let $\mathfrak{T}$ be a subset of the primes such that $T(x)$, the number of $p\leq x$, $p\in\mathfrak{T}$, satisfies $T(x)\sim(\tau x)/\log x$, $\tau>0$, $x\to\infty$. Let $\mathfrak{M}_k(x)$ be the set of natural numbers whose prime factors (a) occur to the $k$th power at least and (b) belong to $\mathfrak{T}$. The author proves that $M_k(x)$, the number of $n\leq x$, $n\in\mathfrak{M}_k$, has the asymptotic formula $M_k(x)\sim(Cx^{1/k})/\log x$, $x\to\infty$, where $C$ depends on $k$ and $\mathfrak{T}$ (in particular, on $\sum_{p\leq x^{1/k},\, p\in\mathfrak{T}}(1/p)$). For $k=1$, this specializes to the result of Wirsing [Arch. Math. **7** (1956), 263–272; MR **18**, 642] and when $\mathfrak{T}$ is the set of all primes ($\tau=1$), to $M_k(x)\sim C^*x^{1/k}$.

*R. D. James* (Vancouver, B.C.)

699:

**Vinogradov, I. M. On an upper bound for $G(n)$.** Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 637–642. (Russian)

The function $G(n)$ is the least number $s$ such that all but a finite number of integers are sums of $s$ positive $n$th powers. The author proved in 1947 [Trav. Inst. Math. Stekloff **23** (1947); MR **10**, 599] that, for $n>2$, $G(n)/n<3\log n+11$. By modifying his method of exponential sums the author shows in the present paper that

$$G(n)/n < 2\log n+4\log\log n+2\log\log\log n+13$$

for all sufficiently large $n$. In particular, one of the lemmas requires $n$ to exceed 170,000.

*D. H. Lehmer* (Berkeley, Calif.)

700:

**Iseki, Shô. A generalization of a functional equation related to the theory of partitions.** Duke Math. J. **27** (1960), 95–110.

The asympotitic theory of partitions is connected with the transformation formula for the Dedekind modular function $\eta(\tau)$. Special types of partition problems make use of transformation formulas analogous to, or more general than, that of Dedekind. One of these was discovered by E. M. Wright [Acta Math. **63** (1934), 143–191] in his study of partitions into $k$th powers. Another generalization of the Dedekind functional equation, recently discovered by the author [same J. **24** (1957), 653–662; MR **19**, 953], has the following form:

$$(1)\quad \sum_{l=0}^{\infty} \{\lambda((l+\alpha)z-i\beta)+\lambda((l+1-\alpha)z+i\beta)\}+z(\alpha^2-\alpha+\tfrac{1}{6})$$
$$=\sum_{l=0}^{\infty} \{\lambda((l+\beta)/z+i\alpha)+\lambda((l+1-\beta)/z-i\alpha)\}$$
$$+(\pi/z)(\beta^2-\beta+\tfrac{1}{6})+2\pi i(\alpha-\tfrac{1}{2})(\beta-\tfrac{1}{2}),$$

where $\lambda(t)=-\log(1-e^{-2\pi t})$, $\Re(z)>0$, $0\leq\alpha\leq 1$, $0<\beta<1$ (or $0<\alpha<1$, $0\leq\beta\leq 1$). In the present paper the author extends this formula in a direction which generalizes Wright's formula. The principal change is that the quantities $(l+\alpha)$ and $(l+1-\alpha)$ which appear on the left of (1) are replaced by their $k$th powers, where $k$ is a positive integer. *T. M. Apostol* (Pasadena, Calif.)

701:

**Nakamura, Yoshio. On the distribution of ideals with exactly $r$ different prime divisors in an ideal class of an algebraic number field.** Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A **6**, 241–251 (1959).

The author extends a result of H.-E. Richert on the distribution of rational integers with exactly $r$ different prime divisors in arithmetical progressions [J. Reine Angew. Math. **192** (1953), 180–203; MR **15**, 603]. The extension is to algebraic number fields $K$ of finite degree $n$, with ideal classes modulo an ideal $m$ in $K$ playing the role of the arithmetical progressions. The author's more general asymptotic estimates depend only upon $K$, $m$ and $r$. The methods, as well as the results, are based directly upon those of Richert and extensively employ the Hecke $L$-functions. *R. Hull* (Santa Monica, Calif.)

702:

**Gel'fond, A. O. A common property of number systems.** Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 809–814. (Russian)

If $\theta>1$ is a real number, then any $\alpha$, $0<\alpha\leq 1$, can be represented in a unique way by the series $\alpha=\sum_1^{\infty}\lambda_n/\theta^n=\sum_1^{n}\lambda_k/\theta^k+x_{n+1}/\theta^n$, $0\leq x_n<1$, where $x_1=\alpha$, $\cdots$, $x_n=x_n(\alpha)=\{\theta x_{n-1}\}$ and $\lambda_1=[\alpha\theta]$, $\cdots$, $\lambda_n=[\theta x_n]$ ($\{x\}$ denotes the proper fractional part of $x$ and $[x]$ the greatest integer $\leq x$). Further, define: (1) $\psi(x)=1$ if $0\leq x\leq 1$, $\psi(x)=0$ if $x<0$ or $x>1$; (2) $1=\sum_1^n\lambda_k/\theta^k+t_{n+1}/\theta^n$, $t_1=1$; (3) $\tau=\sum_1^{\infty}t_k/\theta^{k-1}$; (4) $\sigma_0(t)=\tau^{-1}\sum_1^{\infty}\psi(1-t_k+t)/\theta^{k-1}$, $0<t\leq 1$. The author establishes the law of distribution of the remainders $x_n(\alpha)$ proving that, if $\theta$ is not an integer, then for almost all $\alpha$ there holds

$$\lim_{N\to\infty} N^{-1}\sum_{k=1}^{N}\psi(1-t+x_k)=\int_0^t\sigma_0(x)dx$$
$$=\tau^{-1}\sum_{k=1}^{\infty}\min(t,t_k)/\theta^{k-1}.$$

[The case where $\theta$ is an integer was considered by G. H. Hardy and J. E. Littlewood, Acta Math. **37** (1914), 155–191.] *J. W. Andrushkiw* (Newark, N.J.)

703:

**Lawton, B. A note on well distributed sequences.** Proc. Amer. Math. Soc. **10** (1959), 891–893.

The concept of a well distributed sequence was introduced by Petersen [Quart. J. Math. Oxford Ser. (2) **7** (1956), 188–191; MR **20** #2313a]. The author points out that the analogues of two further theorems about uniform distribution hold for well distribution.

*J. W. S. Cassels* (Cambridge, England)

704:

**Goldberg, Karl. The minimum of a certain linear form.** J. Res. Nat. Bur. Standards Sect. B **64B** (1960), 49–50.

For integers $n\geq 3$, $0<a_1\leq\cdots\leq a_n$, let $L(x)=\sum_{j=1}^{n}a_jx_j$, $M=$the positive minimum of $L(x)$ for all $x_j\geq 0$ such that $L(x)\geq 2a_ix_i$ $(i=1,\cdots,n)$. If either $a_1$ or $a_2$ divides $a_i$, or if $a_i=a_j$ for some $j\neq i$ $(i=3,4,\cdots,n)$ let $r_i=0$; otherwise $r_i=$minimum of the least non-negative remainders (mod $a_1$) of $a_2-a_i$, $2a_2-a_i$, $\cdots$, $[(a_i-1)/a_2]\cdot a_2-a_i$. It is proved that $M=\min\{2[a_1,a_2],2a_3+r_3,2a_4+r_4,\cdots,2a_n+r_n\}$. The general proof is built largely on the case $n=3$ which is settled first.

*H. Schwerdtfeger* (Montreal)

705:

**Watson, G. L. Bounded representations of integers by quadratic forms.** Mathematika **4** (1957), 17–24.

Es sei $f(x_1,\cdots,x_k)$ eine quadratische Form mit ganzzahligen Koeffizienten. Man nennt eine Lösung von (1) $n=f(x_1,\cdots,x_k)$ eine beschränkte Darstellung von $n$ durch $f$, wenn (2) $\max(|x_1|,\cdots,|x_k|)\ll n^{1/2}$ ist. Dabei ist das Zeichen $\ll$ im Sinne von Vinogradov zu verstehen. (D. h. die Ungleichung gilt für eine nicht spezialisierte Konstante auf der rechten Seite.) Es fragt sich, ob man die Konstante unabhängig von $n$ wählen kann. Folgt aus der Lösbarkeit von (1) für beliebiges $n$ auch (2), so sagt man, $f$ stellt ganze Zahlen beschränkt dar. Dies gilt insbesondere für definite Formen. — I. Stellt eine indefinite Form $f$ die Null nicht trivial dar, dann stellt sie ganze Zahlen beschränkt dar, sofern $k\geq 4$ ist. — II. Stellt eine indefinite Form $f$ die Null nur trivial dar, dann stellt sie ganze Zahlen beschränkt dar. — Der Beweis von Satz II benützt die Theorie der Reduktion der positiv definiten Formen, während Satz I aus elementaren Kongruenzen und Ungleichungen folgt. Satz I gilt nicht für $k=2$ und ob er für $k=3$ gilt, ist derzeit noch ungewiß.

*N. Hofreiter* (Zbl **77**, 264)

706:

**Obreškov, N. Sur l'approximation diophantienne des formes linéaires.** Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 2, 3–18 (1959). (Bulgarian. Russian and French summaries)

Résumé de l'auteur: "Nous démontrons les théorèmes suivants. (1) Soit $a$ un nombre positif et entier et $n$ un nombre entier plus grand que $a$. Alors pour chaque nombre

réel $\omega$, $0<\omega\leqq a$, ils existent au moins deux nombres entiers non négatifs $x$ et $y$ tels que l'on a

$$(1)\qquad |\omega x-y| \leqq \left(\left[\frac{n-a}{a+1}\right]+2\right)^{-1}, \quad 0<x+y\leqq n.$$

Le signe d'égalité dans (1) est atteint. (2) Soit $p$ et $d$ deux nombres entiers et positifs, $d\leqq p$, et soit $n>0$ un nombre entier arbitraire mais multiple de $d/(p,d)$. Soit $\omega_1,\cdots,\omega_p$ $p$ nombres réels arbitraires. Alors ils existent $p$ nombres entiers $x_1,\cdots,x_p$, non négatifs, desquels $d$ au moins sont différents de 0 et tels que l'on a ($y$ entier),

$$(2)\qquad |\omega_1x_1+\cdots+\omega_px_p-y| \leqq \frac{1}{(np/d)+1},$$
$$x_s\leqq n \quad (s=1,\cdots,p).$$

Le signe d'égalité dans (2) est atteint."

707:

**Jarník, Vojtěch. Eine Bemerkung zum Übertragungssatz.** Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 2, 169–175 (1959). (Bulgarian and Russian summaries)

Let

$$\Theta=\begin{bmatrix}\theta_{11}&\cdots&\theta_{m1}\\ \cdot&\cdots&\cdot\\ \theta_{1n}&\cdots&\theta_{mn}\end{bmatrix},\qquad H=\begin{bmatrix}\theta_{11}&\cdots&\theta_{1n}\\ \cdot&\cdots&\cdot\\ \theta_{m1}&\cdots&\theta_{mn}\end{bmatrix}=\Theta^T$$

be a real $n\times m$ matrix and its transposed; put

$$S_j(x)=\sum_{\mu=1}^{m}\theta_{\mu j}x_\mu-x_{m+j}\quad(j=1,2,\cdots,n),$$

$$T_i(x)=\sum_{\nu=1}^{n}\theta_{i\nu}x_\nu-x_{n+i}\quad(i=1,2,\cdots,m).$$

Denote by $A(\Theta)$ the least upper bound of all $a>0$ for which there are infinitely many integral solutions $x_1,\cdots,x_m$, $x_{m+1},\cdots,x_{m+n}$ of the inequalities $|S_j(x)|<X^{-a}$ ($j=1,2,\cdots,n$), where $X=\max(|x_1|,\cdots,|x_m|)>0$, and define $A(H)$ analogously. It is known that $m/n\leqq A(\Theta)\leqq\infty$, $n/m\leqq A(H)\leqq\infty$, and a general transfer principle [J. Dyson, Proc. London Math. Soc. (2) **49** (1947), 409–420; MR **9**, 271] states that

$$A(\Theta)\geqq\frac{mA(H)+(m-1)}{(n-1)A(H)+n},$$

with a similar inequality obtained by interchanging $\Theta$ and $H$. The author proves that this transfer principle is best in the following sense. (1) If $n\geqq m\geqq 1$, $n/m\leqq A\leqq\infty$ or $1<n<m$, $(m-1)/(n-1)\leqq A\leqq\infty$, there exists a pair of matrices $\Theta$ and $H$ satisfying

$$(\mathrm{J})\qquad A(H)=A,\quad A(\Theta)=\frac{mA+(m-1)}{(n-1)A+n}.$$

(For the excluded case $1=n<m$ see V. Jarník [Acta Arith. **2** (1936), 1–22].) This result is an immediate consequence of the following one. (2) Let $m\geqq 1$, $n>1$, $\max(n/m,(m-1)/(n-1))\leqq A\leqq\infty$ and let $\Theta$ and $H$ be restricted to the matrices satisfying

$$A(\theta_{11})=A,\quad \theta_{21}=\cdots=\theta_{m1}=0.$$

Then (J) holds for almost all such matrices.

*K. Mahler* (Notre Dame, Ind.)

708:

**Fast, H.; Świerczkowski, S. On the number of lattice points inside a closed curve.** Colloq. Math. **6** (1958), 211–214.

Let $G$ be a plane region and let its Lebesgue measure $|G|$ be an integer. Under certain conditions it is proved that $G$ can be translated and rotated so as to contain exactly $|G|$ lattice points.

*C. G. Lekkerkerker* (Amsterdam)

## FIELDS

See 695.

## LINEAR ALGEBRA

See also 678, 760, B1064, B1070.

709:

**Marcus, Marvin. Basic theorems in matrix theory.** Nat. Bur. Standards Appl. Math. Ser. **57** (1960), iv + 27 pp.

For anyone with even a passing interest in the subject, this is certainly the best book bargain one could imagine! It is a compact collection of definitions, identities, inequalities, and basic theorems, grouped into six chapters, as follows: General definitions and elementary properties; Canonical forms, invariance, congruence, commutativity, orthogonalization; Eigenvalues, determinants, submatrices, rank; Determinant and rank inequalities; Linear equations $Ax=b$; Inversion of $A$; Some particular matrices and their condition numbers. Each of these is well subdivided, so that there is no great difficulty in finding what is there even though the index is perhaps less complete than one might like.

While the collection is to be recommended enthusiastically, it is, nevertheless, not beyond criticism. Statements are often referenced, but by no means always. The list of 47 references is quite meager, and the selection seems a bit haphazard. Localization theorems of the Perron-Gerschgorin-Brauer type are given, but there is no mention of the Weinstein theorem. The Givens method for computing eigenvalues of a hermitian matrix is stated after Jacobi's method, and is called "the Givens' modification", whereas the two methods have only superficial resemblance. The condition for convergence of the Jacobi iteration for solving a system of equations is incorrectly stated (top of p. 18). It is to be hoped, however, that the author will consider subsequent expansions, with appropriate revision.

*A. S. Householder* (Oak Ridge, Tenn.)

710:

**Forsythe, G. E.; Henrici, P. The cyclic Jacobi method for computing the principal values of a complex matrix.** Trans. Amer. Math. Soc. **94** (1960), 1–23.

Jacobi's method for computing the eigenvalues of a real symmetric matrix $A$ uses the iteration $A_0=A$, $A_{k+1}=U_kA_kU_k^T$, where $U_k$ has a $2\times 2$ orthogonal submatrix at the intersection of the $i_k$th and $j_k$th rows and columns and is otherwise the unit matrix. If there exist permutation matrices $P_k$ such that $P_kA_kP_k^{-1}$ tends to a diagonal matrix, the diagonal elements are the eigenvalues of $A$. The authors generalize this method by showing how to

compute the eigenvalues of a hermitian matrix and the principal values of an arbitrary matrix. (The principal values of $A$ are the positive square roots of the eigenvalues of $AA^*$.) Jacobi chose $(i_k, j_k)$ as the position of the element of maximum modulus off the diagonal in $A_k$. The authors show that a simpler method of choosing the sequence $(i_k, j_k)$ $(k=0, 1, 2, \cdots)$—cyclic by rows (or columns)—is admissible. They prove several theorems giving conditions under which the process converges.

*H. S. A. Potter* (Aberdeen)

711:

**Kato, Tosio. Estimation of iterated matrices, with application to the von Neumann condition.** Numer. Math. **2** (1960), 22–29.

For the matrix norm defined by $\|A\| = \max_{u \neq 0} \|Au\|/\|u\|$, the vector norm being Euclidean, the author seeks bounds for $p_n = \|A^n\|/r^n$, where $r$ is the spectral radius of $A$. In particular, he is interested in conditions for $p_n$ to be uniformly bounded over some class of matrices $A$. The method used is to consider those eigenvalues which lie outside some simple, closed, rectifiable curve $\Gamma$ which lies within the circle $|z| \leq r$, and whose distance from the spectrum of $A$ is at least $\alpha > 0$. The expression then depends upon $\alpha, r, \|A\|$, along with the degrees of the elementary divisors for eigenvalues outside $\Gamma$, and the separation distances $\delta_k$ of these eigenvalues, by which is meant the least of the distances of $\lambda_k$ from all eigenvalues distinct from it. No reference is made to papers by Ostrowski [Math. Z. **63** (1955), 2–18; MR **17**, 228] and Gautschi [Duke Math. J. **20** (1953), 127–140, 375–379; Compositio Math. **12** (1954), 1–16; MR **15**, 94; **16**, 105, 326], where, however, closely related problems are considered.

*A. S. Householder* (Oak Ridge, Tenn.)

712:

**Milnes, Harold Willis. A note on bounded continuous matrix products.** Michigan Math. J. **6** (1959), 335–338.

Let $\alpha_1, \alpha_2, \cdots$ be elements of a closed, bounded region of a normed space such that the norms of the differences $\alpha_{i+1} - \alpha_i$ are uniformly bounded, and let $A(\alpha_i)$ be finite matrices such that all eigenvalues satisfy $|\lambda(\alpha_i)| \leq \varepsilon < 1$. If all the matrices $A$ are diagonalizable, then the product $\prod A(\alpha_i)$ is bounded. This is to say that there exists a constant $C$ such that $\|(\prod_{i=1}^{I} A(\alpha_i))x\| \leq C\|x\|$ for every $I$. An example illustrates the relevance of the theorem to the stability of numerical methods of solution of a differential equation.

*A. S. Householder* (Oak Ridge, Tenn.)

713:

**Herz, J.-C. Sur les propriétés spectrales des matrices normales.** Chiffres **2** (1959), 145–146. (English, German and Russian summaries).

Slightly novel proof of unitary diagonalizability.

*A. S. Householder* (Oak Ridge, Tenn.)

714:

**Marcus, Marvin; Khan, Nisar A. On matrix commutators.** Canad. J. Math. **12** (1960), 269–277.

Let $[A, B] = AB - BA$ and $(A, B) = [A, [A, B]]$, where $A$ and $B$ are $n$-square matrices over an algebraically closed field of characteristic 0. There is obtained a formula for the number of linearly independent solutions of $(A, X) = 0$ in terms of the degrees of the elementary divisors of $A$. In addition it is shown that if $[A, X] = 0$ implies $[X, B] = 0$ then $B$ is a scalar polynomial in $A$; moreover, the dimension of the linear space of all such matrices $B$ is the number of distinct eigenvalues of $A$.

*C. R. Putnam* (Lafayette, Ind.)

715:

**Spencer, A. J. M.; Rivlin, R. S. Further results in the theory of matrix polynomials.** Arch. Rational Mech. Anal. **4**, 214–230 (1960).

This paper extends earlier results by the authors [same Arch. **2** (1958/59), 309–336, 435–446; MR **20** #7030, #7031]. It is shown, by making use of the Cayley-Hamilton theorem and some rather intricate computations, that any matrix polynomial $\mathbf{P}$ in $R$ $3 \times 3$ matrices $\mathbf{a}_P$ $(P = 1, \cdots, R)$ can be expressed as a linear combination of matrix products of the forms $\mathbf{I}$, $\mathbf{a}_K$, $\mathbf{a}_K^2$, $\mathbf{a}_K\mathbf{a}_L$, $\mathbf{a}_K\mathbf{a}_L^2$, $\mathbf{a}_K^2\mathbf{a}_L$, $\mathbf{a}_K^2\mathbf{a}_L^2$, $\mathbf{a}_K\mathbf{a}_L\mathbf{a}_K^2$, $\mathbf{a}_K\mathbf{a}_L^2\mathbf{a}_K^2$, $\mathbf{a}_K\mathbf{a}_L\mathbf{a}_M$, $\mathbf{a}_K^2\mathbf{a}_L\mathbf{a}_M$, $\mathbf{a}_K\mathbf{a}_L^2\mathbf{a}_M$, $\mathbf{a}_K\mathbf{a}_L\mathbf{a}_M^2$, $\mathbf{a}_K^2\mathbf{a}_L^2\mathbf{a}_M$, $\mathbf{a}_K\mathbf{a}_L^2\mathbf{a}_M^2$, $\mathbf{a}_K\mathbf{a}_L\mathbf{a}_M\mathbf{a}_K^2$, $\mathbf{a}_K\mathbf{a}_L\mathbf{a}_M\mathbf{a}_N$, $\mathbf{a}_K^2\mathbf{a}_L\mathbf{a}_M\mathbf{a}_N$, $\mathbf{a}_K\mathbf{a}_L^2\mathbf{a}_M\mathbf{a}_N$, $\mathbf{a}_K\mathbf{a}_L\mathbf{a}_M^2\mathbf{a}_N$, $\mathbf{a}_K\mathbf{a}_L\mathbf{a}_M\mathbf{a}_N^2$, $\mathbf{a}_K\mathbf{a}_L\mathbf{a}_M\mathbf{a}_N\mathbf{a}_S$, where $K, L, M, N, S$ are distinct integers from the set $1, \cdots, R$, with coefficients which are scalar polynomial invariants of the matrices. It is stated that certain of these matrix products can be expressed in terms of others, details being promised in a later paper. From these results the authors obtain a basic system for symmetric polynomials in symmetric matrices which has fewer members than that given in the previous papers. The basis for the invariants of a set of five symmetric matrices under the orthogonal group obtained in the previous paper is further reduced. An application to mechanics is indicated.

*J. A. Todd* (Cambridge, England)

716:

**Varga, O. Ein elementargeometrischer Beweis des Sylvester-Frankeschen Determinantensatzes.** Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 2, 105–109 (1959). (Bulgarian and Russian summaries)

An elementary geometric proof of the theorem that for an $n \times n$ matrix $A$,

$$\det C_p(A) = (\det A)^{\binom{n-1}{p-1}},$$

where $C_p(A)$ is the $p$th compound of $A$.

*M. Newman* (Washington, D.C.)

717:

**Asplund, S. O. Finite boundary value problems solved by Green's matrix.** Math. Scand. **7** (1959), 49–56.

This paper discusses the inversion of those non-singular symmetric band matrices having elements $c_{kl} = 0$ for $|k-i| > 1$, and applications.

*P. S. Dwyer* (Ann Arbor, Mich.)

718:

**Asplund, Edgar. Inverses of matrices $\{a_{ij}\}$ which satisfy $a_{ij} = 0$ for $j > i + p$.** Math. Scand. **7** (1959), 57–60.

The property that a non-singular square matrix is triangular if and only if its inverse is triangular is extended to the characterization of invertible matrices whose elements are zero above some other diagonal parallel to the main diagonal. Band matrices of order $p$ and Green's matrices of order $p$ are defined. Two theorems are established which characterize the relations between a non-singular square band matrix and its inverse.

*P. S. Dwyer* (Ann Arbor, Mich.)

## ASSOCIATIVE RINGS AND ALGEBRAS

719:

**Faith, Carl. Rings with minimum condition on principal ideals.** Arch. Math. **10** (1959), 327–330; erratum, 480.

An MP-ring is a ring with minimum condition on principal left ideals. The author's main result is that the Jacobson and (Baer) lower nil radicals coincide for MP-rings. This enables him to prove that a ring $A$ is M-semi-simple (i.e., a direct sum of simple rings having minimal left ideals) if and only if $A$ contains no non-zero nilpotent ideals and is an MP-ring. This latter result is equivalent to a theorem of Kaplansky [Trans. Amer. Math. Soc. **68** (1950), 62–75; MR **11**, 317; Th. 8.1]; and is parallel to the Wedderburn-Artin structure theorem. A number of other properties of MP-rings are given.

*Drury W. Wall* (Iowa City, Iowa)

720:

**Nagata, Masayoshi. On the theory of Henselian rings. III.** Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 93–101.

In this note the author generalizes his earlier results on Henselizations and their properties [e.g., Nagoya Math. J. **7** (1954), 1–19; MR **16**, 788] to arbitrary quasi-local rings $\mathfrak{o}$. Suppose that $R$ is a normal quasi-local ring containing an ideal $\mathfrak{a}$ for which $\mathfrak{o}=R/\mathfrak{a}$. Then, $R^*$ denoting the Henselization of $R$ [loc. cit.], $R^*/\mathfrak{a}R^*=\mathfrak{o}^*$ is called a Henselization of $\mathfrak{o}$. One of the author's theorems states that $\mathfrak{o}^*$ is unique within isomorphisms [theorem 1 stating that $\mathfrak{o}^*$ is Henselian, and theorem 3 affirming the uniqueness]. Furthermore, the author investigates the relations between two concepts of unramified extensions $\mathfrak{o}'/\mathfrak{o}$, with respective maximal ideals in $\mathfrak{m}'$, $\mathfrak{m}$. ($U_1$) $\mathfrak{m}'=\mathfrak{o}'\mathfrak{m}$, $\mathfrak{o}'/\mathfrak{m}'$ separable over $\mathfrak{o}/\mathfrak{m}$; ($U_2$) the Henselization of $\mathfrak{o}'$ is a separable integral extension of the Henselization of $\mathfrak{o}$. For example (theorem 8), $U_1$ is equivalent to $U_2$, provided the derived normal ring of $\mathfrak{o}$ is quasi-local, assuming furthermore that $\mathfrak{o}'$ is a ring of quotients of a ring which is a finite module over $\mathfrak{o}$. The importance of the last assumption is emphasized by means of several examples. The paper also contains a correction to a chain of lemmas, loc. cit.

*O. F. G. Schilling* (Chicago, Ill.)

721:

**Nagahara, Takasi; Onodera, Takesi; Tominaga, Hisao. On normal basis theorem and strictly Galois extensions.** Math. J. Okayama Univ. **8** (1958), 133–142.

Let $R$ denote a simple ring with minimum condition, $G$ a group of automorphisms of $R$, and $S$ the fixed ring. $V_R(S)$ denotes the centralizer of $S$ in $R$. Both $S$ and $V_R(S)$ are assumed simple with minimum condition. The authors augment the results of Kasch on the "normal basis theorem" [Math. Ann. **126** (1953), 447–463; MR **15**, 597] by proving that his sufficient condition is also necessary: $V_R(S)=V_R(R)$ or $V_R(S)\subset S$. In another direction, the authors show that if $R$ has characteristic $p\neq 0$, if $G$ is a finite $p$-group, and if the inner automorphisms in $G$ are produced by elements of $R$ which generate a subring without zero divisors or two-sided ideals, then $V_R(S)$ is commutative, generated by the centers of $R$ and $S$. If, besides, $R$ is a division ring (so that the hypotheses on $G$, except $p$-power order, are automatic) then every subring of $R$ containing $S$ is generated over $S$ by a single element. If, furthermore, $S$ is perfect then $R$ is commutative. There are also some remarks on decomposability of group rings over simple rings.

*D. Zelinsky* (Evanston, Ill.)

## NON-ASSOCIATIVE RINGS AND ALGEBRAS

722:

**Ögmundsson, Vilhjálmur. Multiplication in $n$ dimensions.** Nordisk Mat. Tidskr. **7** (1959), 111–116, 144.

The author gives an elementary proof of Hurwitz's result that the only finite-dimensional linear algebras over the reals having a norm with $\|xy\|=\|x\|\,\|y\|$ are the real and complex numbers, the quaternions, and the Cayley numbers.

*J. B. Crabtree* (Hoboken, N.J.)

723:

**Schafer, R. D. On cubic forms permitting composition.** Proc. Amer. Math. Soc. **10** (1959), 917–925.

It is known that a finite-dimensional separable alternative algebra $A$ over $F$ has the form $A=A_1+\cdots+A_r$, where each $A_i$ is central simple of dimension $m_i^2$ over its center $Z_i$, $Z_i$ is a separable extension of $F$ of degree $d_i$, and the principal norm $n_i(x_i)$ for $x_i$ in $A_i$ is homogeneous of degree $m_id_i$ satisfying $n_i(x_iy_i)=n_i(x_i)n_i(y_i)$. A mapping $x\to N(x)$ of a vector space over a field $F$ of characteristic $\neq 2, 3$ is called a cubic form on $V$ in case $N(\alpha x)=\alpha^3N(x)$ for all $\alpha$ in $F$ and $x$ in $V$ and $(x, y, z)=\frac{1}{6}[N(x+y+z)-N(x+y)-N(x+z)-N(y+z)+N(x)+N(y)+N(z)]$ is trilinear. The trilinear form $(x, y, z)$ and its associated cubic form $N(x)$ are non-degenerate if $(x, y, z)=0$ for all $y, z$ in $V$ implies $x=0$. It is proved that if $A$ is a finite-dimensional non-associative algebra with 1 over $F$ of characteristic $\neq 2, 3$, a necessary and sufficient condition for the existence of a non-degenerate cubic form $N(x)$ on $A$ permitting composition is that $A$ be a separable alternative algebra for which $N(x)$ is given by $N(x)=[n_1(x_1)^{f_1}]\cdots[n_r(x_r)^{f_r}]$ of degree $\sum_{i=1}^{r}f_im_id_i=3$. That is, $A$ is one of (i) $F1$ (with $f_1=3$), (ii) a cubic field over $F$, (iii) a central simple associative algebra of dimension 9 over $F$, (iv) $Fe_1+B$, where $B$ is an algebra (with unity element $1-e_1$) on which a non-degenerate quadratic form permitting composition is defined. The possible dimensions for $A$ are 1, 2, 3, 5, 9. For proof it is first shown that $A$ is power-associative. Then it is shown that $B(x, y)=T(xy)$, where $T(x)=3(x, 1, 1)$, is an admissible trace function for $A$. Albert's results on trace-admissible algebras [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 317–322; MR **11**, 6] restrict the possibilities for $A$, and by considering the possible cases the main theorem is obtained.

*L. A. Kokoris* (Chicago, Ill.)

724:

**Ritchie, R. W. A generalization of non-commutative Jordan algebras.** Proc. Amer. Math. Soc. **10** (1959), 926–930.

A generalization of the non-commutative Jordan algebras is obtained by considering the class of algebras satisfying $(x^2y)x=x^2(yx)$ and $x^2x=xx^2$. Here the weaker identity $x^2x=xx^2$ replaces the flexible identity $(xy)x=x(yx)$ of non-commutative Jordan algebras. It is proved

that every finite-dimensional semi-simple algebra of characteristic not 2 or 3 belonging to this generalized class is a non-commutative Jordan algebra with identity element. Earlier results on non-commutative Jordan algebras [see Oehmke, Ann. of Math. (2) **68** (1958), 221–230; MR **21** #5664] then imply that a semi-simple algebra is a direct sum of simple algebras and that a simple algebra is either a (commutative) Jordan algebra, a quasi-associative algebra, or an algebra of degree 1 or 2.

*L. A. Kokoris* (Chicago, Ill.)

## HOMOLOGICAL ALGEBRA

725:

**Matlis, Eben. Injective modules over Prüfer rings.** Nagoya Math. J. **15** (1959), 57–69.

Let $R$ be an integral domain with quotient field $Q$. An $R$-module $A$ is said to be linearly compact if every finitely solvable set of congruences $x \equiv x_\alpha \pmod{A_\alpha}$, with submodules $A_\alpha$ of $A$ and $x_\alpha \in A_\alpha$, has a simultaneous solution in $A$, while $A$ is called semi-compact or pre-compact if the solvability of the above congruences holds under the additional condition that $A_\alpha$ are annihilators of ideals of $R$ or the intersection $\bigcap A_\alpha$ is non-zero respectively. It is shown that every injective $R$-module is semi-compact (and divisible), and conversely if $R$ is a Prüfer ring every divisible semi-compact $R$-module is injective. Every $R$-homomorphic image of $Q$ is injective if, and only if, $R$ is a Prüfer ring and every $R$-homomorphic image of $Q$ is semi-compact. Now $R$ is called a valuation ring if every element of $Q$ not in $R$ is an inverse of an element of $R$; further, a valuation ring $R$ is said to be maximal or almost maximal if $Q$, as an $R$-module, is linearly compact or pre-compact respectively. The author then proves that a valuation ring $R$ is almost maximal if and only if $K = Q/R$ is injective, and an almost maximal valuation ring $R$ is maximal if and only if $\mathrm{Hom}_R(K, K) \cong R$. If $R$ is a maximal valuation ring, there exists a unique indecomposable injective module $E$ with a one-to-one correspondence between ideals of $R$ and submodules of $E$ by means of the annihilator relation. Furthermore, maximal valuation rings may be characterized as those Prüfer rings with linearly compact quotient field, or as those $R$ such that $\mathrm{Ext}_R^1(A, S) = 0$ for any torsion-free module $A$ and any torsion-free module $S$ of rank one. Finally, it is proved that a necessary and sufficient condition for a Noetherian $R$ to be a Dedekind ring is that every finitely generated indecomposable torsion $R$-module is cyclic.

*G. Azumaya* (Sapporo)

726:

**Sonner, Johann. On limits of module-systems.** Proc. Amer. Math. Soc. **10** (1959), 983–986.

The author defines, on a pair $(E, M)$ of sets, a "structure $(f, g)$ of a module-system over a ring $R$", which is reminiscent of a sheaf. For each filter $\mathfrak{F}$ on $M$, a "limit" module, $\lim_{\mathfrak{F}}(f, g)$, is then defined, in terms of germs of sections over the elements of $\mathfrak{F}$. The result elegantly reformulates the definition of "limit" given by the reviewer in Proc. Amer. Math. Soc. **9** (1958), 118–129 [MR **20** #1712], where a relation $R$ on $M$ was included for the application, but was unnecessary for the definition itself. The author then considers the questions of maps, and of induced structures; and introduces a notion of cofinality, relative to $(f, g)$, of one filter $\mathfrak{F}'$ on $M$ with respect to another, $\mathfrak{F}$, on $M$. Using these notions, a theorem is proved, saying when $\lim_{\mathfrak{F}}(f, g)$ is isomorphic with $\lim_{\mathfrak{F}'}(f', g')$, given structures $(f, g)$, $(f', g')$ on $(E, M)$.

*H. B. Griffiths* (Birmingham)

## GROUPS AND GENERALIZATIONS

727:

**Kurosh, A. G. ★The theory of groups.** Translated from the Russian and edited by K. A. Hirsch. 2nd English ed. 2 volumes. Chelsea Publishing Co., New York, 1960. Vol. 1: 272 pp. $4.95; Vol. 2: 308 pp. $4.95.

Differs from first English edition [1955, 1956; MR **17**, 124; **18**, 188] in a number of small changes and corrections.

728:

**Conrad, Paul. The groups of word preserving permutations of a group.** J. Indian Math. Soc. (N.S.) **22** (1958), 149–179 (1959).

Let $G$ be a given group and $f$ any group word. Denote by $[f]$ the group of permutations on $G$ which preserve $f$, e.g., $[XY]$ is the automorphism group of $G$ and $[XY^{-1}Z]$ is the holomorph of $G$. In this paper various groups $[f]$ are studied by elementary methods.

The longest section is concerned with displaying the groups $[X^n]$ as direct products of wreath products (both unrestricted) in a natural way by means of the sets of imprimitivity $G^{\mathfrak{w}} = \{h \mid h \in G \text{ and } h^{w^r} = g^{w^s} \text{ for some } r, s \geq 0\}$. The direct decompositions of $[X^{-1}]$ are then studied in detail. Typical of the many other results are: (1) If $G$ is not of order 3, then the following are equivalent: (a) $x^{n+1} = e$ for all $x$ in $G$; (b) $[X^n] = [X^{-1}]$; (c) $[X^{n+1}] = [X^0]$; (d) $[X^{n+2}] = [X]$. (2) If $[XYX^{-1}Y^{-1}] \supseteq [X^0]$ and $[XY^{-1}Z] = [XY]T$, where $T$ is a cyclic normal subgroup of $[XY^{-1}Z]$ with $T \cap [XY] = 1$, then $G \cong T$. *P. J. Higgins* (London)

729:

**Parker, E. T. On quadruply transitive groups.** Pacific J. Math. **9** (1959), 829–836.

The only known finite permutation groups that are (at least) 4-fold transitive are the alternating groups $A_n$ for $n > 5$, the symmetric groups $S_n$ for $n > 3$, and the Mathieu groups of degrees $n = 11$, 12, 23, and 24. The author imposes various conditions on a Sylow subgroup of the subgroup $H$ fixing four symbols in a quadruply transitive group $G$, conditions that are then shown to ensure that $G$ is an alternating or symmetric group. The simplest such conditions are those of Theorem 2, namely that the degree $n$ of $G$ and an odd prime divisor $p$ of the order of $H$ satisfy the inequalities $4p \neq n - 4 < 5p$. In the dissertation [Ohio State University, 1957] on which the paper is based, the author has also presented a self-contained proof that a quadruply transitive permutation group of a type other than those known must have degree exceeding 27; but this proof is not included in the paper on the grounds that it is long, and that the result is largely contained (though scattered) in the literature.

*B. H. Neumann* (Manchester)

730:

**Leptin, Horst. Abelsche $p$-Gruppen und ihre Automorphismengruppen.** Math. Z. **73** (1960), 235–253.

Let $\Gamma(G)$ denote the automorphism group of the group $G$. If $G$ and $G'$ are abelian $p$-groups with $p>3$ and $\Gamma(G)$ is isomorphic to $\Gamma(G')$ then $G$ and $G'$ are isomorphic. If $G$ is an abelian $p$-group and $G'$ an abelian $q$-group with $q>p>3$ then $\Gamma(G)$ is isomorphic to $\Gamma(G')$ only if $G$ and $G'$ are cyclic. The proofs depend on known direct sum decompositions of abelian $p$-groups. These decompositions of course are associated with idempotent endomorphisms of $G$ or, equivalently, with involutions ($e^2=e$ if and only if $u=2e-1$ satisfies $u^2=1$). An isomorphism of $\Gamma(G)$ to $\Gamma(G')$ sends commuting involutions to commuting involutions, thus carrying a decomposition of $G$ into a decomposition of $G'$. The proof that the number of summands of any order is the same in both decompositions requires a finer analysis. {Reviewer's note: In view of the results in section 19 of Kaplansky, *Infinite abelian groups* [Univ. of Michigan Press, Ann Arbor, 1954; MR **16**, 444], the first theorem above can be interpreted as follows: The endomorphism ring of an abelian $p$-group, $p>3$, is uniquely determined by its group of units. However, not every isomorphism of such unit groups is extendable to an isomorphism of endomorphism rings, nor is it induced by an isomorphism of the groups operated on.}

*D. Zelinsky* (Berkeley, Calif.)

731:

**Brauer, Richard; Suzuki, Michio. On finite groups of even order whose 2-Sylow group is a quaternion group.** Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1757–1759.

Let $G$ be a finite group whose 2-Sylow group $P$ is a quaternion group (ordinary or generalized). Then, it is proved, $G$ is not simple. More precisely, if $H$ denotes the maximal normal subgroup of odd order, $G/H$ has a center of order 2. The proof, which is merely sketched in the present paper, makes use of the character relation $\sum_i \chi_i(\mu)^2\chi_i(\sigma)/\chi_i(1)=0$ for an element $\sigma$ of even order in $G$ and the unique involution $\mu$ in $P$; cf. Brauer and Fowler, Ann. of Math. (2) **62** (1955), 565–583 [MR **17**, 580] for the general formula, of which this is a special case. On expressing $\chi_i(\sigma)$ by (generalized) decomposition numbers and modular characters, and on letting $\sigma$ range over so-called 2-sections of certain elements of $P$, the authors derive a relation of type $\sum_i \chi_i(\mu)^2 t_i/\chi_i(1)=0$ ($t_i=0, \pm 1$), which is further combined with the properties of decomposition numbers, such as orthogonalities, to prove $\chi_i(\mu)=\chi_i(1)$ for some $i$, whence $G$ is not simple; the case of the ordinary quaternion group $P$ is treated under the additional condition that $G$ has no normal subgroup of index 2, and is very complicated, while the case of a generalized quaternion group $P$ is (only) comparatively simpler.

*T. Nakayama* (Nagoya)

732:

**Kozel, P. T. Maximum soluble subgroups of the full linear group $GL(n, D)$ over the field of real numbers.** Dokl. Akad. Nauk BSSR **2** (1958), 279–282. (Russian)

A primitive group $\Gamma$ of the type mentioned in the title can be of two types. (a) The maximal abelian normal divisor $F$ of $\Gamma$ is the multiplicative group of reals. (b) $F$ is isomorphic to the multiplicative group of complex numbers. In the first case, $\Gamma$ has a solvable subgroup $A$ of index 72, which contains $F$ properly. Generating relations for $\Gamma/A$, $A/F$ are given. In the second case, $\Gamma$ consists of matrices of the form $\operatorname{diag}(L, L)$, $L=\begin{pmatrix}\lambda & -\mu\\ \mu & \lambda\end{pmatrix}$, $\lambda^2+\mu^2>0$.

If $\Gamma$ is imprimitive, it may have 2 or 4 systems of imprimitivity. In the latter case, $\Gamma$ is a semidirect product of the (complete) diagonal group by a group isomorphic with the symmetric group of order 4. In the former case, $\Gamma=TG$, where $T$ is $\operatorname{diag}(Q, Q)$, $Q=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, and $G$ is $\{\operatorname{diag}(G', E_2), \operatorname{diag}(E_2, G')\}$, $G'$ a maximal primitive solvable subgroup (of degree 2).

The proofs are only sketched. Reliance is placed on the reference Suprunenko, Belorussk. Gos. Univ. Uč. Zap. **12** (1951), 74–112. See also Suprunenko, Dokl. Akad. Nauk SSSR **83** (1952), 183–186 [MR **14**, 14].

*J. L. Brenner* (Palo Alto, Calif.)

733:

**Takahashi, Shuichi. Arithmetic of group representations.** Tôhoku Math. J. (2) **11** (1959), 216–246.

This paper gives a complete account of all the basic facts known on integral representations. Its bibliography makes it possible to track down very easily virtually all the important literature on the subject to date. The author did not learn of the two fundamental papers of Maranda [Canad. J. Math. **5** (1953), 344–355; **7** (1955), 516–526; MR **15**, 100; **19**, 529] until his investigation was nearly complete, and a greater part of his work consists in an independent reconstruction of Maranda's results. In view of this, a detailed review would be inappropriate; we shall merely indicate some of the new ideas. For representations over Dedekind rings, the author uses Chevalley's version of the Steinitz elementary divisor theory, which is an improvement over the methods of Maranda's second paper. Zassenhaus' binding systems (Verbindungssysteme) have been replaced by appropriate cohomological devices complete with a "principal genus theorem". It has long been known that something homological was going on here but, to the reviewer's knowledge, this is the first time anybody has taken the trouble to write it all down. The slight amount of homology theory necessary is developed ab ovo so that the uninitiate need not be frightened away. The most interesting new idea in the paper is concerned with studying the classes of integral representations in a given genus. This is done by extending the representations to the adèle ring of the ground field and constructing a certain algebraic group, namely the group of all regular endomorphisms commuting with the extended representation. A one-to-one correspondence is constructed between the classes in a given genus and a particular family of double cosets of this algebraic group. This gives the relations between the local and global class numbers of the representations and the absolute ideal class number of the field; in particular, it gives the relations of this type that were obtained by Maranda. The theory of integral representations has perhaps not yet justified itself on the score of new group-theoretical results. However it has already produced some new invariants and there are high hopes for things to come. This paper is recommended, not only for its lucid account of what is already known, but also for its interesting and stimulating ideas suggestive of what the future may hold.

*W. E. Jenner* (Lewisburg, Pa.)

734:

**Terehov, A. A. Completely orderable groups.** Dokl. Akad. Nauk SSSR **129** (1959), 34–36. (Russian)

Eine Gruppe $G$ heisst $V$-Gruppe, wenn jede Totalordnung einer beliebigen Untergruppe $A \subset G$ zu einer Totalordnung von $G$ erweitert werden kann. In der Arbeit sind die folgenden Sätze bewiesen: Eine nilpotente $V$-Gruppe ist abelsch. Eine auflösbare $V$-Gruppe hat die Höhe 2 und besitzt einen abelschen Normalteiler $A$ derart, dass die Faktorgruppe $G/A$ mit einer Untergruppe der multiplikativen Gruppe von positiven rationalen Zahlen isomorph ist. Eine lokal auflösbare $V$-Gruppe ist eine auflösbare Gruppe der Höhe 2. Sind alle Untergruppen mit endlich vielen Erzeugenden einer Gruppe $G$ $V$-Gruppen, so ist auch $G$ eine $V$-Gruppe.

*J. Jakubík* (Košice)

735:

**Conrad, Paul. Ordered semigroups.** Nagoya Math. J. **16** (1960), 51–64.

An ordered semigroup is said to be positive if $x+x>x$. A criterion is given for a semigroup to be the ordinal sum of positive ordered semigroups. It is proved that any positive ordered semigroup is order-homomorphic to an ordinal sum of semigroups each order-isomorphic to a semigroup of positive real numbers; and that if a semigroup is embedded in a group by Ore's method [Ann. of Math. (2) **32** (1931), 463–477], then the extension of the order to the group is unique.

*H. A. Thurston* (Vancouver, B.C.)

736:

**Zareckiĭ, K. A. On ideals of semigroups.** Uspehi Mat. Nauk **14** (1959), no. 6 (90), 173–174. (Russian)

Let $C_M$ denote the semigroup of all transformations of the set $M$ into itself and for cardinals $\xi$, not exceeding the cardinality of $M$, let $C_M{}^\xi$ denote the members of $C_M$ whose images have cardinality less than $\xi$. The author proves the following. Let $A$ be a semigroup (having more than one element) and $B$ be an ideal of $A$. Then there is a set $M$ and an isomorphism $\phi$ of $A$ into $C_M$ such that $\phi(B)=\phi(A)\cap C_M{}^\xi$ for some $\xi$.

*K. A. Ross* (Seattle, Wash.)

## TOPOLOGICAL GROUPS AND LIE THEORY

737:

**Mickle, E. J.; Radó, T. A uniqueness theorem for Haar measure.** Trans. Amer. Math. Soc. **93** (1959), 492–508.

The authors observe that the literature on measure theory is incomplete in several respects, and that, in particular, Haar measures should be studied in spaces in which the topology, rather than the distance, is invariant under the relevant group of homeomorphisms. The uniqueness theorem which they obtain is designed with a view to specific applications in integral geometry; these are illustrated by the case of the space of oriented lines in Euclidean 3-space. The proof depends on background material which again has to be developed so as to supplement gaps in the available literature. The statement of the uniqueness theorem involves two conditions, one of which is perhaps a little surprising, and the presence of these conditions, which are satisfied in the applications that the authors have in mind, suggests that a general study of Haar measures, in the spirit of the paper, will be far from trivial.

*L. C. Young* (Madison, Wis.)

738:

**Sternberg, Shlomo. The structure of local homeomorphisms. III.** Amer. J. Math. **81** (1959), 578–604.

Let $V^n$ [$U^n$] be the group of all local $C^\infty$ volume-preserving [respectively, volume-preserving up to a constant factor] mappings in $n$-space which leave the origin fixed; $H^n$ the group of all local Hamiltonian mappings in $2n$-space; $C^n$ the set of all mappings given by $x_i^*=x_i f_i(\omega)$, $\omega=x_1\cdots x_n$, with $\prod f_i=1$ (which is proved to be a maximal commutative subgroup of $V^n$). Theorems: 1–4. If $T\in V^n$, $n>2$, has a diagonizable Jacobian with eigenvalues $\lambda_i$ such that if $\lambda_i=\lambda_1^{m_1}\cdots\lambda_n^{m_n}$, then $m_i-1=m_j$, $j\neq i$, there exists $R\in V^n$ such that $RTR^{-1}\in C^n$; if $n=2$ the same holds true under the additional assumption $|\lambda_i|\neq 1$; each such $T$ lies in a one-parameter group in $V^n$. 5–7. If $T\in U^n$ and $\lambda_i\neq\lambda_1^{m_1}\cdots\lambda_n^{m_n}$ for all $m_j\geq 0$, $\sum m_j>1$, there exists $R\in V^n$ such that $RTR^{-1}$ is the linear part of $T$; every such $T$ lies in a one-parameter group in $U^n$. 9. If $T\in H^n$, $|\lambda_i|\neq 1$ and $\lambda_1^{m_1}\cdots\lambda_n^{m_n}\neq 1$ for all integers $m_j$ positive or negative which are not all zero, there exists an $R\in H^n$ such that $RTR^{-1}$ is of the form $x_i^*=x_i f_i(\omega_1,\cdots,\omega_n)$, $y_i^*=y_i g_i(\omega_1,\cdots,\omega_n)$, $\omega_i=x_i y_i$, $f_i g_i=1$, $\partial\log f_i/\partial\omega_j=\partial\log f_j/\partial\omega_i$. The proofs are via corresponding theorems for formal mappings. References: by the author [same J. **79** (1957), 809–824; **80** (1958), 623–631; MR **20** #3335, #3336]; J. Moser [Comm. Pure Appl. Math. **9** (1956), 673–692; MR **19**, 278]; C. L. Siegel [ibid. **10** (1957), 305–309; MR **19**, 1048].

*J. L. Massera* (Montevideo)

739:

**Matsushima, Yozô. Espaces homogènes de Stein des groupes de Lie complexes.** Nagoya Math. J. **16** (1960), 205–218.

Considering complex Lie groups and complex homogeneous spaces, the author studies their structure as complex manifolds. The main results are as follows. (1) Let $G$ be a connected complex Lie group with Lie algebra $\mathfrak{g}$, and $K$ a maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$. $\mathfrak{k}$ is a real subalgebra of $\mathfrak{g}$, and $\tilde{\mathfrak{k}}=\mathfrak{k}+(\sqrt{-1})\mathfrak{k}$ is a complex subalgebra of $\mathfrak{g}$. Let $\tilde{K}$ denote the connected complex Lie subgroup of $G$ corresponding to $\tilde{\mathfrak{k}}$. Then, as a complex manifold, $G$ is holomorphically homeomorphic to $\tilde{K}\times C^n$, where $C^n$ denotes complex $n$-space. (2) Let $G$ and $K$ be as in (1). Suppose that $G$ is the complex form of $K$, namely, that $\mathfrak{k}+(\sqrt{-1})\mathfrak{k}=\mathfrak{g}$ ($\tilde{K}=G$) and $\mathfrak{k}\cap(\sqrt{-1})\mathfrak{k}=0$ with the above notation. Then $G$ is a complex affine algebraic group. (3) A complex Lie group $G$ is called a Stein group if $G$, as a complex manifold, is a Stein manifold. A connected Stein group is, then, a complex affine algebraic variety. (4) Let $G$ be a Stein group and $H$ a closed connected complex subgroup of $G$. If $H$ is the complex form of its maximal compact subgroup, then the homogeneous space $G/H$ is a Stein manifold. (5) Let $G$ be a connected complex semi-simple Lie group and $H$ a closed connected complex subgroup of $G$. If $G/H$ is a Stein manifold, then $H$ is the complex form of its maximal compact subgroup and $G/H$ is a complex affine algebraic variety.

*K. Iwasawa* (Cambridge, Mass.)

740:

**Kirillov, A. A. Induced representations of nilpotent Lie groups.** Dokl. Akad. Nauk SSSR **128** (1959), 886–889. (Russian)

The author shows how to obtain all irreducible unitary representations of nilpotent Lie groups as induced representations. The result (whose proof is sketched) is as follows. Let $\mathfrak{G}$ be a connected nilpotent Lie group. Then every irreducible unitary representation of $\mathfrak{G}$ is induced by a one-dimensional unitary representation of some subgroup. More precisely, there is a system $(\mathfrak{G}_\alpha)$ of subgroups of $\mathfrak{G}$, with corresponding sets of one-dimensional unitary representations ('admissible' representations), such that (i) every admissible representation induces an irreducible representation of $\mathfrak{G}$, (ii) every irreducible representation of $\mathfrak{G}$ is induced by some admissible representation, (iii) equivalent irreducible representations of $\mathfrak{G}$ are induced by equivalent admissible representations of the same subgroup, (iv) every $\mathfrak{G}_\alpha$ contains the centre of $\mathfrak{G}$. Further, for each $\mathfrak{G}_\alpha$, the admissible representations are given explicitly as representations of a certain abelian homomorphic image of $\mathfrak{G}_\alpha$. *P. M. Cohn* (Manchester)

741:

**Dixmier, Jacques. Sur l'algèbre enveloppante d'une algèbre de Lie nilpotente.** Arch. Math. **10** (1959), 321–326.

Let $\mathfrak{g}$ be a Lie algebra (of finite dimension over a field $K$ of characteristic zero throughout) and denote by $\mathfrak{Z}(\mathfrak{g})$ the centre of the universal associative enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$. The author proves two theorems about $\mathfrak{Z}(\mathfrak{g})$ in case $\mathfrak{g}$ is nilpotent. Theorem 1: $\mathfrak{Z}(\mathfrak{g})$ is isomorphic (as algebra) to the subalgebra $\mathfrak{I}(\mathfrak{g})$ of the symmetric algebra $\mathfrak{S}(\mathfrak{g})$ on $\mathfrak{g}$ consisting of the invariant elements (i.e., elements mapped to zero by the derivations induced on $\mathfrak{S}(\mathfrak{g})$ by elements of $\mathfrak{g}$). In fact the canonical (vector space-) isomorphism $\varphi: \mathfrak{U}(\mathfrak{g}) \to \mathfrak{S}(\mathfrak{g})$ [cf. Bourbaki, *Groupes et algèbres de Lie*, Ch. 1, Actualités Sci. Ind. no. 1285, Hermann, Paris, 1960; p. 34] which clearly maps $\mathfrak{Z}(\mathfrak{g})$ onto $\mathfrak{I}(\mathfrak{g})$ is shown to be an algebra isomorphism when $\mathfrak{g}$ is nilpotent (examples show that this is false for general $\mathfrak{g}$). As a corollary it is proved that (for nilpotent $\mathfrak{g}$) $\mathfrak{Z}(\mathfrak{g})$ is a factorial ring (=integral domain with unique prime factorization).—Theorem 2: Let $\mathfrak{g}$ be a nilpotent Lie algebra with centre $\mathfrak{c}$; then for all 1-dimensional subspaces $\mathfrak{c}'$ of $\mathfrak{c}$ except a finite number the following holds: If $\theta: \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}/\mathfrak{c}')$ is the canonical mapping, then $\theta(\mathfrak{Z}(\mathfrak{g}))$ and $\mathfrak{Z}(\mathfrak{g}/\mathfrak{c}')$ have the same field of fractions, and if the fields generated by $\mathfrak{Z}(\mathfrak{g})$, $\mathfrak{Z}(\mathfrak{g}/\mathfrak{c}')$ have transcendence degrees $\delta$, $\delta'$ respectively over $K$, then $\delta' = \delta - 1$.

*P. M. Cohn* (Manchester)

## MISCELLANEOUS TOPOLOGICAL ALGEBRA

742:

**Hofmann, Karl Heinrich. Topologische distributive Doppelloops.** Math. Z. **71** (1959), 36–68.

In dieser Arbeit werden die früheren Untersuchungen des Verf. [insbesondere Math. Z. **70** (1958), 213–230; MR **21** #1364] fortgesetzt. Sie ist jedoch im wesentlichen unabhängig davon lesbar. Unter einer Doppelloop wird eine additive Loop $K$ mit Neutralelement 0 verstanden, deren von 0 verschiedene Elemente eine multiplikative Loop $K'$ bilden, so daß $0x = x0 = 0$ für alle $x$ gilt. Solche Strukturen treten als Koordinatenbereich projektiver Ebenen auf. Eine distributive Doppelloop mit assoziativer Multiplikation heißt Neokörper, insbesondere reeller oder komplexer Neokörper, wenn die multiplikative Gruppe isomorph ist zur multiplikativen Gruppe der reellen oder komplexen Zahlen. Es werden ausschließlich lokal kompakte topologische Doppelloops betrachtet. Für diese werden unter anderem folgende Strukturaussagen gewonnen, welche die früheren Ergebnisse teilweise verbessern: (1) Ist die multiplikative Gruppe einer lokal kompakten, zusammenhängenden, endlich dimensionalen Doppelloop eine kommutative Gruppe, so ist sie zur multiplikativen Gruppe des Körpers der reellen oder komplexen Zahlen isomorph. (2) Eine lokal kompakte, zusammenhängende, endlich dimensionale, distributive Doppelloop hat einen reellen oder komplexen Neokörper als Zentrum. Eine weniger einfache Aussage wird im nulldimensionalen Fall gemacht. (3) Eine lokal zu einem reellen Intervall homöomorphe, einseitig distributive Doppelloop ist ein reeller Neokörper. (4) Eine lokal zu einer Kreisscheibe homöomorphe, entweder einseitig distributive Doppelloop, in der je zwei Elemente in einer multiplikativen Gruppe liegen, oder beidseitig distributive Doppelloop, in der jedes Element in einer multiplikativen Halbgruppe liegt, ist ein komplexer Neokörper. (5) Eine lokal kompakte, zusammenhängende, endlich dimensionale distributive Doppelloop, deren Elementepaare in additiven Gruppen liegen, hat den reellen oder komplexen Körper oder den Quaternionenkörper als Kern.

Für die Formulierung weiterer Ergebnisse, insbesondere bei Dimension 0, sowie des Hauptsatzes, und eine Übersicht über die reellen Neokörper muß auf die Arbeit selbst verwiesen werden. {Bemerkung des Ref.: Die Behauptungen (1) und (2) sind auch ohne die Voraussetzung endlicher Dimension richtig.}

*H. Salzmann* (Frankfurt a.M.)

743:

**Hofmann, Karl Heinrich. Topologische Doppelloops und topologische Halbgruppen.** Math. Ann. **138** (1959), 239–258.

Verf. betrachtet in der vorliegenden Arbeit ausschließlich lokal kompakte, zusammenhängende topologische Doppelloops mit assoziativer Multiplikation, im folgenden kurz "Doppelloops" genannt [vgl. #742]. Eine topologische Halbgruppe nennt Verf. punktierbar, wenn sie als topologischer Raum homogen ist und durch Weglassen eines geeigneten Punktes zu einer topologischen Gruppe wird. Eine Doppelloop bildet bezüglich der Multiplikation stets eine solche punktierbare Halbgruppe. Satz 1: Eine lokal kompakte, zusammenhängende, punktierbare topologische Halbgruppe ist isomorph zur multiplikativen Halbgruppe der reellen oder komplexen Zahlen oder der Quaternionen. {Bemerkung des Ref.: Verf. beweist Satz 1 nur unter der zusätzlichen Voraussetzung endlicher Dimension und gibt in einem Satz 2 eine Beschreibung der Halbgruppen bei unendlicher Dimension. Durch Kombination der von ihm selbst entwickelten Methoden mit denen des Ref. [Math. Z. **67** (1957), 436–466; MR **21** #2950; § 8] läßt sich aber die vom Verf. als offen hingestellte Frage nach der Existenz solcher unendlich dimensionaler Halbgruppen negativ beantworten.} Insbesondere ist die

multiplikative Gruppe einer Doppelloop isomorph zur multiplikativen Gruppe der reellen oder komplexen Zahlen oder der Quaternionen. {Auch hier betrachtet Verf. noch den unendlich-dimensionalen Fall, der aber wiederum nicht auftreten kann.} Der letzte (vierdimensionale) Fall wird bei zusätzlicher Voraussetzung eines Distributivgesetzes und der Potenzassoziativität der Addition weiter untersucht; vgl. die Arbeit selbst. Ist die Addition sogar alternativ, so ist sie kommutativ. Es gibt kontinuierlich viele solcher einseitig distributiven Doppelloops mit alternativer Addition. Sie enthalten genau einen komplexen Unterkörper, wenn sie vom Quaternionenkörper verschieden sind.

*H. Salzmann* (Frankfurt a.M.)

## FUNCTIONS OF REAL VARIABLES

744:

**Lederer, G. A problem on Baire classes.** Fund. Math. **48** (1959), 85–89.

Given a closed subset $F$ of $[0, 1]$, a function $f(x)$ has property $D(\alpha)$ with respect to $F$ at $x_0$ if, for any $\varepsilon>0$, there is an open neighbourhood $I$ of $x_0$ and a function $g(x)$ of Baire class $\leq\alpha$ such that $|f(x)-g(x)|<\varepsilon$ a.e. over $F\cap I$. The author proves that if $\alpha>0$, and $f(x)$ is such that for each non-empty closed $F\subset[0, 1]$, $f(x)$ has property $D(\alpha)$ with respect to $F$ at least at one point $x_0$ in $F$, then there exists a function $g(x)$ of Baire class $\leq\alpha$, such that $f(x)=g(x)$ a.e. The case $\alpha=1$ of this result enables the author to give an affirmative answer to a question raised by Marczewski. Thus if $f(x)$ is such that for every non-empty closed $F\subset[0, 1]$ there is an $x_0\in F$ such that $f(x)$ is almost continuous with respect to $F$ at $x_0$, then there is a function $g(x)$ of Baire class $\leq 1$ such that $f(x)=g(x)$ a.e. in $[0, 1]$. *S. J. Taylor* (Ithaca, N.Y.)

745:

**Hammer, Preston C.; Holladay, John C. Functions of restricted variation.** Rend. Circ. Nat. Palermo (2) **8** (1959), 102–114.

Let $S$ be a simply ordered set and $f$ a real or complex function on $S$. By limiting to $k$ the number of points in a partition, the authors define a $k$-variation $V(k, f)$, analogous to total variation, concerning which various curious identities and inequalities are established. The function $f$ is said to be of restricted variation if $V(k+1, f)-V(k, f)=0$. Various conditions turn out to be equivalent to this, for instance, the existence of an ordered set containing $S$, on which $f$ has a continuous extension.

*L. C. Young* (Madison, Wis.)

746:

**De Vito, Luciano. Sulle funzioni di più variabili a variazione limitata.** Arch. Rational Mech. Anal. **3** (1959), 60–81.

Dato il punto $P=(x_1, \cdots, x_r)\in S^{(r)}$ e l'intero $h$, con $1\leq h\leq r$, sia $\alpha^{(h)}$ una combinazione di classe $h$ degli indici $1, \cdots, r$ $(\alpha^{(h)}=a_1<a_2\cdots<a_h)$, e sia $b_1<\cdots<b_{r-h}$ la $(r-h)$-pla formata con gli indici residui. Posto $P=(\xi, \eta)$, con $\xi=(x_{a_1}, \cdots, x_{a_h})$, $\eta=(x_{b_1}, \cdots, x_{b_{r-h}})$, sia $I$ un intervallo (superiormente aperto) di $S^{(r)}$, $I_\xi$ e $I_\eta$ le proiezioni sugli spazi coordinati $S^{(\xi)}$, $S^{(\eta)}$; poniamo infine $a_j=k$, $x_k=t$. Una funzione $u(P)$ si dice a variazione limitata secondo Tonelli rispetto ad $x_k$, $[\mathrm{VL}]_{a^{(h)},k}$, se tra le funzioni ad essa equivalenti ne esiste una (che indicheremo ancora con $u$) tale che risulti: (a) per ogni $\eta\in I_\eta$, $u(t, \eta)$ è VL in $I_\xi$; (b) la corrispondente variazione $V_\eta$ è maggiorabile da una funzione misurabile e sommabile in $I_\eta$.

Fichera ha dato la seguente definizione di funzione a variazione limitata $[\mathrm{VL}]_{a^{(h)},k}$: (a') posto, per ogni $J_\xi\subset I_\xi$,

$$\Phi_{a^{(h)}}(J_\xi, \eta)=\int_{FJ_\xi} u(\xi, \eta)\nu_t d\omega$$

(ove $\nu_t$ è la componente secondo l'asse $t$ del versore normale esterno a $FJ_\xi$), la $\Phi_{a^{(h)}}$, come funzione dell'intervallo $J_\xi$, per ogni fissato $\eta$, è VL in $I_\xi$; (b') la variazione totale $V_{\Phi_{a^{(h)}}}(\eta)$ è maggiorata da una funzione misurabile e sommabile in $I_\eta$. In questo lavoro, l'A. dimostra l'equivalenza delle due definizioni (già provata da Fichera per $r=2$, $u$ continua); posto inoltre $[\partial u/\partial t]_{a^{(h)},k}=\Phi'_{a^{(h)}}$, dimostra che tutte le derivate ottenibili, per un dato $k$, differiscono tra loro, al più, in un insieme di misura nulla.

*L. Amerio* (Milan)

747:

**Bagley, R. W.; McKnight, J. D. Differentiation of infinite series of functions.** Amer. Math. Monthly **66** (1959), 890–892.

Soit $\{f_n(x)\}$ une suite de fonctions réelles de la variable réelle $x$, définies sur un segment $[a, b]$, et convergeant vers $f(x)$; on dit qu'elle converge uniformément en $x_0$ si, à tout $\varepsilon>0$, on peut faire correspondre un $\delta>0$, et un entier $N$ tels que: $|f_n(x)-f(x)|<\varepsilon$, pour $|x-x_0|<\delta$, $n\geq N$. L'auteur démontre le théorème suivant: si $f_n(x)$ a une dérivée continue sur $[a, b]$, si $f_n(x)\to f(x)$, pour tout $x$, et si $f_n'(x)$ converge vers $g(x)$ uniformément en $x_0$, alors $f'(x_0)$ existe et vaut $g(x_0)$. De plus, si $\{f_n'\}$ converge vers $g(x)$ sur $[a, b]$, alors dans tout segment $[c, d]\subset[a, n]$, il y a un $x_0$ où $f'(x_0)$ existe et vaut $g(x_0)$. Il s'ensuit que les points où $f(x)$ n'a pas de dérivée, ou en a une différente de $g(x)$, forment un ensemble de première catégorie.

*J. Favard* (Paris)

748:

**Urbanik, K. On a problem of S. L. Cheng concerning sequences of functions with convergent $k$-th differences.** Ann. Polon. Math. **7** (1959), 33–40.

Pour une fonction $f(x)$, réelle, mesurable sur $(-\infty, +\infty)$, on pose:

$$\Delta_h^k f(x)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(x+jh).$$

En désignant par $\{f_n\}$ une suite de réelles fonctions, l'hypothèse: $\lim_{n\to\infty}\Delta_h^k f_n(x)=0$ pour tout $h$, uniformément par rapport à $x$ dans tout intervalle fini, est équivalente aux égalités: $f_n(x)=\sum_0^{k-1}a_{jn}x^j+g_n(x)$ où les $a_{jn}$ sont des constantes, et où la suite $g_n$ tend uniformément vers 0 dans tout intervalle fini. *J. Favard* (Paris)

749:

**Bellman, Richard. On inequalities with alternating signs.** Proc. Amer. Math. Soc. **10** (1959), 807–809.

Let $a_1\geq a_2\geq\cdots\geq a_m\geq 0$, $1\geq w_1\geq w_2\geq\cdots\geq w_m\geq 0$, and let $h(x)$ be a convex function with $h(0)\leq 0$. I. Olkin [Proc.

Nat. Acad. Sci. U.S.A. **45** (1959), 230–231; addendum, 1553; MR **21** #3528] proved the inequality

$$(1)\qquad \sum_{j=1}^{m}(-1)^{j-1}w_j h(a_j) \geqq h\Big(\sum_{j=1}^{m}(-1)^{j-1}w_j a_j\Big).$$

This is a generalization of an inequality given by the author [Amer. Math. Monthly **60** (1953), 402; MR **14**, 957], which, in turn, generalizes an inequality given by the reviewer [Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 611–613; MR **14**, 24]. (The latter two inequalities are already contained in an earlier work of G. Szegö [Math. Z. **52** (1950), 676–685; MR **12**, 703; cf. p. 682].)

The author points out that (1) is an immediate consequence of the inequality

$$(2)\qquad \int_a^b f(t)g(t)dt \leqq \int_a^{a+c} f(t)dt$$

where $f$ is non-increasing, $0 \leqq g(t) \leqq 1$, and $c = \int_a^b g dt$, given by J. F. Steffensen [C. R. Dixième Congrès Math. Scandinaves 1946, pp. 181–186, Jul. Gjellerups Forlag, Copenhagen, 1947; MR **8**, 457]. He then generalizes (2) to

$$(3)\qquad \left(\int_a^b f(t)g(t)dt\right)^p \leqq \int_a^{a+c} f^p dt$$

where $f$ is non-negative and non-increasing, $g \geqq 0$, $\int_a^b g^{p'}dt \leqq 1$, $p>1$, $1/p+1/p'=1$, and $c=(\int_a^b g dt)^p$. {There is a misprint in the definition of $c$.}

*H. F. Weinberger* (Minneapolis, Minn.)

750:

**Mandelbrojt, Szolem. Sur un problème de Gelfand et Shilov.** C. R. Acad. Sci. Paris **249** (1959), 2144–2145.

Soient $M_n$ et $N_n$ deux suites $\geq 0$; $S=S(M_n, N_n)$ désigne l'ensemble des fonctions $f$ indéfiniment différentiables sur $R$ pour lesquelles il existe $A$, $h$ et $k$ tels que $|x^m f^{(n)}(x)| \leq Ak^n M_n h^m N_m$, pour tout $n$ et $m$. Le problème de Gel'fand et Šilov, posé dans le tome II de leur ouvrage *Obobščennye funkcii* [Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958; MR **21** #5142a], est: indiquer des conditions sur $M_n$ et $N_n$ pour que $S$ ne soit pas vide (réduit à la fonction nulle).

Soit $\mu_n = M_{n-1}/M_n$, $n \geq 1$, $\mu_0 = M_0 = 1$. Supposons que $\sum \mu_n^2 < \infty$, et introduisons $\mathfrak{R}(y) = y\sum_{\mu_n y>1}\mu_n + y^2\sum_{\mu_n y \leq 1}\mu_n^2$, $A(x) = \sup_{y\geq 0}(xy - \mathfrak{R}(y))$, $x>0$.

L'A. montre alors (par un procédé explicite de construction de $f$) qu'il existe une fonction $f$, non identiquement nulle, telle que $|f^{(n)}(x)| \leq M_n \exp(-A(|x|))$, pour tout $n$. Si l'on note que la classe $S$ est identique à celle des fonctions $f$ vérifiant $|f^{(n)}(x)| \leq Ak^n M_n \exp(-L(|x|))$, où $L(x) = \sup_m (m\log x - \log N_m)$, $x>0$, on en déduit que la classe $S$ n'est pas vide dès que $A(x) \geqq L(x)$, $x>0$. Ce théorème contient comme cas particulier un résultat de Šilov [Dokl. Akad. Nauk SSSR **102** (1955), 893–895; MR **17**, 351] relatif au cas $M_n = n^{\alpha n}$, $N_n = n^{\beta n}$, $\alpha+\beta \geq 1$.

*J. L. Lions* (Nancy)

751:

**Mandelbrojt, Szolem. Sur les classes de Gelfand et Shilov.** C. R. Acad. Sci. Paris **249** (1959), 2465–2466.

Notations du résumé précédent. La suite $N_n$ est dite duale de $M_n$ si $S(M_n, N_n)$ n'est pas vide. Alors si $\sum \mu_n^2 < \infty$, l'A. montre que la suite $N_n$ définie par $N_n = n!/(\nu(n))^n$, $n \geq 1$, $N_0 = 1$ (où $\nu(x)$ est la fonction inverse de $\mathfrak{R}$), est duale de $M_n$. L'A. annonce ensuite divers résultats relatifs aux problèmes correspondants sur la demi droite $[1, \infty[$.

*J. L. Lions* (Nancy)

## MEASURE AND INTEGRATION

752a:

**Denjoy, Arnaud. La métrique de Stieltjes.** C. R. Acad. Sci. Paris **249** (1959), 2437–2442.

752b:

**Denjoy, Arnaud. L'intégrale de Stieltjes.** C. R. Acad. Sci. Paris **250** (1960), 23–29.

752c:

**Denjoy, Arnaud. Les fonctionnelles linéaires.** C. R. Acad. Sci. Paris **250** (1960), 250–255.

These three papers are in a way a supplement to the part of Chapter VIII of the author's *Leçons sur le calcul des coefficients d'une série trigonométrique* [Gauthier-Villars, Paris, 1949; MR **11**, 99; pp. 392–465] entitled "Intégrales et totales quelconques". The underlying space $U$ is usually assumed to be an $r$-dimensional Euclidean space. Ideas presented revolve around the fact that a measure $M$ defines an integral $I$ defines a linear functional $W$, and on the other hand, a linear functional $W_0$ defines a measure $M$ defines an integral $I$ which is a linear functional $W$ and may be more general than $W_0$. The first paper centers on measure notions, suggests a procedure similar to totalisation for determining the positive and negative parts of an unbounded measure, developes the usual theory of measurability based on a point function, which by differences leads to an interval function, the basic intervals being assumed to be open on the right or semi-segment inferior (s.s.i.): $a_i \leq x_i < b_i$, $i=1, \cdots, r$. Sufficient conditions under which a set of measure functions $\nu(E, t)$ depending on a parameter $t$, bounded on subsets of an s.s.i. $[CD)$, with positive and negative values allowed, have a continuous (in $t$) derivative are given. The second paper, with a poke at Heaviside and Dirac, is given over largely to Stieltjes integrals (for Riemann-Stieltjes integrals, basic subdivisions are s.s.i., convergence as maximum length of subdivision approaches zero). Sufficient conditions for derivation under the integral sign of $\int F(T, X)d\nu(T)$, with respect to $X$, are applied to integrals of the form $\int F(T-X)d\nu(T)$ on the linear interval. The third paper is devoted to linear functionals $W(f; E)$ assumed to be defined on subsets $E$ of a class of sets $\Phi(E)$ and a class of functions $G(f)$, additive in $E$, linear in $f$. Distinction is made between synthetic functionals depending on $f$ throughout $E$, and analytic functionals which are essentially point functions $F(X)$ depending on $f$ in the vicinity of $X$, as in the case of derivatives. If the functional is defined on all s.s.i., then $W(f; E)$ obviously determines the point function $F(X)$ by the condition $F(X) = \pm W(f; [OX))$, where $O$ is the origin, the sign depending on the location of $X$ relative to $O$. On the other hand, $f$ in a way can be determined at a point $X$ (as an analytic functional) by the values $F(s) = W(f; s)$, $s$ ranging over the s.s.i. containing $X$. Under conditions on the functions $F$, the character of $f$ can be made rather precise. It is suggested that analytic functionals can be written as limits of synthetic functionals or integrals, by noting for example that if $f(x)$ is continuous at $x$ on $[a, b]$ then $f(x) = \lim_{y\to 0}(1/\pi)\int_a^b f(t)g(t-x, y)dt$, with $g(u, y) = (1/2i)(1/(u-iy) - 1/(u+iy))$, leading to similar expressions for derivatives.

*T. A. Hildebrandt* (Ann Arbor, Mich.)

753:

**Skvorcov, V. A. Interrelation between Denjoy's general integral and totalization $(T_{2s})_0$.** Dokl. Akad. Nauk SSSR **127** (1959), 975–976. (Russian)

Each of the following four generalizations of the Lebesgue theory assigns an integral to any everywhere convergent trigonometric series: the totalization $(T_{2s})_0$ of Denjoy, the $P^2$-integral of P. James, the SCP-integral of Burkill, and the integral of Marcinkiewicz-Zygmund. It is easy to see that each of these integrals includes the Denjoy integral in the restricted sense. The author gives an example showing that the four integrals do not fulfil Lusin's condition (N) and are not compatible with the Denjoy-Khintchine integral in the wide sense (two integrals $J_1$, $J_2$ being termed compatible if $J_1(f; I) = J_2(f; I)$ whenever $f$ is both $J_1$- and $J_2$-integrable on the interval $I$). *M. Cotlar* (Buenos Aires)

754:

**Vinokurov, V. G. Generalized Lebesgue spaces.** Dokl. Akad. Nauk SSSR **129** (1959), 9–11. (Russian)

Un espace $E$ sur lequel est définie une mesure complète $m$ est appelé espace de Lebesgue généralisé si: (1) $m(E)=1$; (2) quels que soient $x, y \in E$, il existe un ensemble mesurable $A \subset E$ tel que $x \in A$ et $y \notin A$; (3) quel que soit la partition mesurable $\Phi$ de $E$, l'espace quotient $E/\Phi$ est un espace de Lebesgue au sens de V. A. Rohlin [Mat. Sb. (N.S.) **25 (67)** (1949), 107–150; MR **11**, 18]. En utilisant une autre définition de l'espace de Lebesgue généralisé, l'A. énonce quelques propriétés de cet espace. Il considère aussi quelques propriétés des processus aléatoires définis dans l'espace de Lebesgue généralisé.

*N. Dinculeanu* (Bucharest)

755:

**Haupt, Otto; Pauc, Christian Y. Über eine Erweiterung des Fubinischen Satzes.** Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1959**, 1–41.

If $Z^*$ is an abstract set and $w|\mathfrak{w}$ is a function on a class $\mathfrak{w}$ of subsets $W$ of $Z^*$, then $w|\mathfrak{w}$ is defined as a complete measure if for all $W$ of $\mathfrak{w}$: $0 \leq w(W) \leq +\infty$; $w$ is $\sigma$-additive; $\mathfrak{w}$ is a $\sigma$-ring of sets; and a set $M$ of $Z^*$ is in $\mathfrak{w}$ if and only if for all $\epsilon > 0$, there exists $W' \leq M \leq W''$ such that $w(W'' - W') < \epsilon$. If $X^*$ and $Y^*$ are abstract sets and $u|\mathfrak{x}$ and $v|\mathfrak{y}$ are complete measures in $X^*$ and $Y^*$ respectively, then the finite-valued measures $u_s|\mathfrak{x}_s$ and $v_s|\mathfrak{y}_s$ define a $j|\mathfrak{q}$ relative to $Z^* = X^* \times Y^*$ by the condition: $Z$ is in $\mathfrak{q}$ if $Z = X \times Y$, where $X$ is in $\mathfrak{x}_s$ and $Y$ in $\mathfrak{y}_s$, and $j(Z) = u(X)v(Y)$. $j|\mathfrak{q}$ can be extended to $j'|\mathfrak{q}'$, the smallest $\sigma$-ring extension of $j|\mathfrak{q}$, and to $j'^{\mathfrak{v}}|\mathfrak{q}'^{\mathfrak{v}}$, the smallest complete measure containing $j|\mathfrak{q}$. The sets of $\mathfrak{q}'^{\mathfrak{v}}$ are $\sigma$-finite relative to $j$. On $j'^{\mathfrak{v}}|\mathfrak{q}'^{\mathfrak{v}}$ the Fubini theorem reads: If $f(x, y)$ is on $P = X \times Y$, with $X$ in $\mathfrak{x}_s'$ (the $\sigma$-ring extension of $\mathfrak{x}_s$) and $Y$ in $\mathfrak{y}_s'$, is measurable relative to $\mathfrak{q}'^{\mathfrak{v}}$, and is $j'^{\mathfrak{v}}$ integrable, then $\int_P f dj'^{\mathfrak{v}} = \int_X (\int_Y f dv) du = \int_Y (\int_X f du) dv$, where the inner integrals exist except for $x$ in a null set $N(u)$ of $X$ and $y$ in a null set $N(v)$ of $Y$. Integrability here is in terms of successive subdivisions and values are on the extended line: $-\infty \leq a \leq +\infty$. To extend this theorem the notion of nilmenge is introduced by the condition: a set $N$ of $Z^* = X^* \times Y^*$ is a nilmenge if for the characteristic function $c(x, y; N)$ of $N$ and every $X$ in $\mathfrak{x}_s$ and $Y$ in $\mathfrak{y}_s$, we have

$$\int_X \left( \int_Y c(x, y; N) dv(y) \right) du(x) = \int_Y \left( \int_X c(x, y; N) du(x) \right) dv(y) = 0,$$

all integrals existing. The class of nilmenge $n_i(u, v)$ is a $\sigma$-ideal in the subsets of $Z^*$ and an extension of $n(j'^{\mathfrak{v}})$, the class of null sets of $\mathfrak{q}'^{\mathfrak{v}}$. By adjoining the class $n_i(u, v)$ of nilmenge to the class $j|\mathfrak{q}$ [see O. Haupt and C. Y. Pauc, same Abh. **1957**, 273–290; MR **20** #2417] to obtain the extension $j^0|\mathfrak{q}^0$, then the $\sigma$-ring extension of the latter: $j^{0'}|\mathfrak{q}^{0'}$ is a complete measure. The extended Fubini theorem results if one replaces $j'^{\mathfrak{v}}|\mathfrak{q}'^{\mathfrak{v}}$ by $j^{0'}|\mathfrak{q}^{0'}$ in the hypothesis. This theorem is compared with one due to W. W. Bledsoe and A. P. Morse [Trans. Amer. Math. Soc. **79** (1955), 173–215; MR **16**, 1008]. In the setting of the present paper a nilset as defined by Bledsoe and Morse is a set $N$ such that

$$\int_{X^*} \left( \int_{Y^*} c(x, y; N) dv(y) \right) du(x) = \int_{Y^*} \left( \int_{X^*} c(x, y; N) du(x) \right) dv(y) = 0,$$

integrals being in the Bledsoe-Morse sense which requires that for fixed $x$ the set of $y$ for which $c(x, y; N) = 1$ is a subset of the $\sigma$-ring extension $\mathfrak{y}_s'$ of $\mathfrak{y}_s$, and similarly for fixed $y$. The class of nilsets $n_{BM}(u, v)$ is a subclass of the class $n_i(u, v)$ of nilmenge. By adjoining $n_{BM}(u, v)$ to $j|\mathfrak{q}$ and extending to the complete measure $j_{BM}{}^{0'}|\mathfrak{q}_{BM}{}^{0'}$, it is shown that the Bledsoe-Morse form of the Fubini theorem is equivalent to replacing $j'^{\mathfrak{v}}|\mathfrak{q}'^{\mathfrak{v}}$ by $j_{BM}{}^{0'}|\mathfrak{q}_{BM}{}^{0'}$ and is included in that obtained in this paper. The last section of the paper considers singular sets. If $Z^*$ is an abstract set, $w|\mathfrak{w}$ is a measure on sets of $Z^*$, then $S$ is $w$-singular if $S$ is hereditarily measurable (every subset $S'$ of $S$ is in $\mathfrak{w}$) and every subset of $S$ has $w$-measure zero or $+\infty$, with $w(S) = +\infty$. Such sets were considered by G. Choquet [Bull. Soc. Math. France **74** (1946), 1–14, 15–25; MR **9**, 419, 275]. Sufficient for a singular set is: If $w|\mathfrak{w}$ is a Carathéodory regular measure (if the set $Z$ is not $w$-measurable, there exists a $W$ of finite $w$-measure such that $Z \cap W$ is not $w$-measurable); if the ideal $\mathfrak{w}_s$ of sets of finite measure in $\mathfrak{w}$ contains a majorant system $\mathfrak{m}$ of power $\aleph_1$ which does not contain the empty set; if the single-element sets are $w$-null sets; and if the set $E$ is contained in no $M$ of $\mathfrak{w}$; then $E$ contains a $w$-singular set $S$. There are sufficient conditions that a set $M$ be non-singular, and in particular $\sigma$-finite relative to $w$.

*T. H. Hildebrandt* (Ann Arbor, Mich.)

756:

**Král, Josef. Note on strong generalized Jacobians.** Czechoslovak Math. J. **9 (84)** (1959), 429–439. (Russian summary)

In this paper the author gives examples which answer in the negative questions concerning plane mappings and the corresponding "strong" (or crude) concepts of bounded variation (sBV), absolute continuity (sAC), multiplicity function ($N_s$), and generalized Jacobian ($J_s$) [see T. Radó, *Length and area*, Amer. Math. Soc., New York, 1948; MR **9**, 505]. If $T: z = T(w)$ is a mapping from a (simply

connected) Jordan region $A$ of the $w$-plane into the $z$-plane $E_2$, and $D$ is a bounded domain of the $w$-plane, then $N_s = N_s(z, T, D)$ denotes the number of the points of the set $T^{-1}(z) \cap D$, and $T$ is said to be sAC in $D$ provided $M \subset D$, meas $M = 0$ implies meas $T(M) = 0$. One of the examples of the present paper shows that there exists a mapping $T$ which is sBV and sAC in some bounded domain $D$, and for which the basic relation $(D)$ $\iint J_s dw = (E_2) \iint N_s dz$ does not hold. {T. Radó [loc. cit.] and L. Cesari [*Surface area*, Princeton Univ. Press, Princeton, N.J., 1956; MR 17, 596] replaced the above "strong" concepts by "weak" or refined concepts which have been shown to be adequate in the theory of transformation of double integrals, in area theory, surface integrals, and applications.} *L. Cesari* (Ann Arbor, Mich.)

## FUNCTIONS OF COMPLEX VARIABLES

See also 792, 809, 848, 901, 904.

757:

**Povolockiĭ, A. I. Indices of singular points of pseudo-analytic functions.** Dokl. Akad. Nauk SSSR **129** (1959), 265–267. (Russian)

Generalization of a theorem of Gahov and Krikunov [Izv. Akad. Nauk SSSR Ser. Mat. **20** (1956), 207–240; MR **19**, 540] concerning indices of singular points of pseudo-analytic functions [cf. M. Morse, *Topological methods in the theory of functions of a complex variable*, Ann. of Math. Studies, no. 15, Princeton Univ. Press, Princeton, N.J., 1947; MR **9**, 20]. The following theorem is established: Let $f$ be a pseudo-analytic function save for a finite number of exceptional points in an $n$-tuply connected region $D$ and let $I$ denote the vector index of Re $f$ with respect to the boundary of $D$. Then

$$k + \sum_{j=1}^{l} (m_j + 1) + \sum_{j=1}^{s} (n_j + 1) - \mu = 2 - \nu + I,$$

where $k$ = the number of places in $D$ at which $f$ has a logarithmic ramification point, $l$ = number of poles, $m_j$ = the order of the $j$th pole, $s$ = number of points $a_j$ in the neighborhood of which $f$ admits a representation of the form

$$f(z) = \frac{1}{(z-a_j)^{n_j}} + A_j \log (z - a_j) + F_j(z)$$

($n_j$ positive integer, $A_j$ real number, $F_j$ analytic at $a_j$), $\mu$ = number of ramification point antecedents counted according to order of ramification.

*M. H. Heins* (Urbana, Ill.)

758:

**Pommerenke, Chr. On some problems by Erdös, Herzog and Piranian.** Michigan Math. J. **6** (1959), 221–225.

Let $f(z)$ be a polynomial with highest coefficient 1, let $E$ denote the point set defined by $|f(z)| < 1$, and $C$ be the boundary of $E$, that is, the lemniscate $|f(z)| = 1$. The author answers a number of questions concerning $E$ and $C$ which were proposed in a paper by P. Erdös, F. Herzog and G. Piranian [J. Analyse Math. **6** (1958), 125–148; MR **21** #123]. Most of the questions treated by the author in the present paper concern the case in which $E$ is connected. Under this assumption it is shown that the length of $C$ is greater than $2\pi$ (except when $f(z) = z^n$), and that $C$ lies in the circle $|z - \zeta| < 2$, where $\zeta$ is the centroid of the zeros of $f(z)$; also upper and lower bounds for the diameter and the width of $E$ are given. Use is made in these results of the fact that the inverse of the function $w = [f(z)]^{1/n}$ maps $|w| > 1$ univalently on the exterior of $C$.

{Editor's note: The following correction has been received from the author. "As F. Herzog kindly pointed out to me the reasoning that leads to $b^2 + d^2 \leq 64/3$ is incorrect. Since $b^2 \leq 32/3$, only $b^2 + d^2 \leq 32/3 + 16 = 80/3$ follows. Corresponding changes apply to $b + d$ and $bd$."}

*F. Herzog* (East Lansing, Mich.)

759:

**Pommerenke, Ch. On some metric properties of polynomials with real zeros.** Michigan Math. J. **6** (1959), 377–380.

Let $f(z)$ be a polynomial with leading coefficient 1 and real zeros, and let $E$ denote the set $|f(z)| \leq 1$. The author shows that $E$ is the union of orthogonal circles, that is, circles with center on the real axis. This result, together with the fact that the transfinite diameter of $E$ is 1, is used to obtain upper estimates for the sum of the diameters of the components of $E$ and for the minimal width of $E$. (Compare the author's paper reviewed above, treating the case of complex zeros, but assuming $E$ to be connected.)

*F. Herzog* (East Lansing, Mich.)

760:

**Derwidué, L. Sur la localisation des zéros des polynomes à coefficients complexes.** Mathesis **68** (1959), 13–23.

The author in a previous study [Mathesis **67** (1958), 214–217; MR **21** #1380] gave a theorem which was an extension of that of Frank [Amer. Math. Monthly **59** (1952), 300–309; Math. Nachr. **19** (1958), 182–185; MR **14**, 6; **21** #2728]. In this paper the author states his theorem more precisely, as follows. Let

$$f_j \equiv x^{s_j}(1 + q_j x^2 + p_j x^4 + \cdots + a_j x^{2r_j}),$$

$$k_j \equiv U_j + Q_j x^2 + P_j x^4 + \cdots + A_j x^{2\sigma_j}$$

$$(j = 0, 1, \cdots, n-1),$$

where $s_j$ is odd and the coefficients $q_j$, $p_j$, $\cdots$, $a_j$, $U_j$, $Q_j$, $P_j$, $\cdots$, $A_j$, and $c_j$ are real. The equation

$$H_n(x) \equiv \begin{vmatrix} c_0 f_0 + k_0 i + 1 & -1 & 0 & & & 0 \\ 1 & c_1 f_1 + k_1 i & -1 & & & \\ & & \ddots & & & \\ & & & 1 & c_{n-2} f_{n-2} + k_{n-2} i & -1 \\ 0 & & & & 1 & c_{n-1} f_{n-1} + k_{n-1} \end{vmatrix} = 0$$

has no pure imaginary roots, and has the same number of roots with positive real parts as the equation

$$H_n(x, 1) \equiv \begin{vmatrix} c_0 f_0 + 1 & -1 & 0 & & & 0 \\ 1 & c_1 f_1 & -1 & & & \\ & & \ddots & & & \\ & & & 1 & c_{n-2} f_{n-2} & -1 \\ 0 & & & & 1 & c_{n-1} f_{n-1} \end{vmatrix} = 0.$$

Furthermore, the equation

$$\Phi_{2n+1}(x, 1) \equiv \begin{vmatrix} c_0 f_0 + k_0 i + 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & c_1 f_1 + k_1 i & -1 & 0 & 0 & 0 \\ 1 & 1 & c_2 f_2 + k_2 i & 0 & 0 & \\ 0 & 0 & 0 & c_3 f_3 + k_3 i & -1 & \\ & & & & \cdots & -1 \\ 1 & 0 & & & 1 & c_{2n} f_{2n} + k_{2n} i \end{vmatrix} = 0$$

(which is assumed to have no pure imaginary roots) has roots with positive real parts in number equal to $v_0 + v_1 + \cdots + v_{2n} + \mu$, augmented by an even number. This even number is zero if $s_j = 1$, $t_j = 0$. Here $t_j$ is the even number of pure imaginary zeros of $f_j$; $v_j$ is the number of zeros of $f_j$ with positive real parts and $v_0$ the number for $f_0 + k_0 i + 1$; $\mu$ is the number of negative $c_j$. As applications, a method is given for the calculation of the roots of algebraic equations with complex coefficients analogous to that given by Hitchcock [J. Math. Phys. Mass. Inst. Tech. **18** (1939), 202–210; MR **1**, 31] and by Frank [Bull. Amer. Math. Soc. **52** (1946), 144–158, 890–898; MR **8**, 154], as well as a simple proof of a necessary and sufficient condition of Frank for all the roots of an equation with complex coefficients to have negative real parts.

*E. Frank* (Chicago, Ill.)

761:

**Lunc, G. L. A remark on functions defined by certain Dirichlet series.** Uspehi Mat. Nauk **14** (1959), no. 6 (90), 175–178. (Russian)

The author proves that some results obtained by N. Pandey [Bull. Calcutta Math. Soc. **47** (1955), 211–215; MR **18**, 200] are valid for a larger class of functions. Denote $H(s) = \sum_{n=1}^{\infty} \phi(n) e^{-sf(n)}$ and $G(s) = \int_0^{\infty} \phi(x) e^{-sf(x)} dx$, and let the functions $f(z)$ and $\phi(z)$ ($z = x + iy$) have the following properties: (1) they are analytic in an angle with the vertex at $z = 0$ containing the positive real axis (and at $z = 0$), (2) $f(z) = o(z)$, (3) $\lim_{z \to \infty} |\ln z / f(z)| < \infty$, (4) $\operatorname{Re} f(z) \geqq 0$ for large enough $|z|$, (5) $|\arg f(x)| < \phi_0 < \frac{1}{2}\pi$ for large enough $x$, (6) $\phi(z) = O(e^{k|f(z)| + l|y|})$, $l < 2\pi$, $k$ a real constant; then $H(s) - G(s)$ represents an entire function. It is also shown that under some conditions $H(s)$ is an entire function, in particular, if $f(z)$ and $\phi(z)$ satisfy the conditions (1)–(6) and $l < 0$, $H(s)$ (and $G(s)$) is an entire function.

*J. W. Andrushkiw* (Newark, N.J.)

762:

**Iliev, L. Eine Bedingung für die Nichtfortsetzbarkeit der Potenzreihen.** Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 2, 205–211 (1959). (Bulgarian. Russian and German summaries)

La note représente la continuation des recherches de l'auteur relatives au prolongement analytique des fonctions définies par les séries de Taylor, qui généralisent le théorème classique de Szegö [L. Bieberbach, *Analytische Fortsetzung*, Springer, Berlin, 1955; MR **16**, 913; p. 120]. Soit $E(\alpha)$ la classe des fonctions $\sum_0^{\infty} c_n z^n$ avec $|c_n| = O(n^{\alpha})$, $\alpha$ réel, arbitraire, et $\{b_n\}$ tel que $b_n / b_{n+1} \to 1$, $0 < l \leqq |b_n| / n^{\alpha} \leqq L$. Soit (*) $\sum d_n b_n z^n \in E(\alpha)$ avec $\{d_n\}$ borné, s'il existe alors une suite $n_1, n_2, \cdots, n_\nu, \cdots$ telle que $\lim d_{n_\nu - k} = \lim d_{n_{\nu+1} - k}$ $(\nu \to \infty)$, $k = 1, 2, \cdots, p_\nu$, avec $p_{n_\nu} \to \infty$ $(n_\nu \to \infty)$ et $|d_{n_\nu} - d_{n_{\nu+1}}| \geqq d > 0$, alors $|z| = 1$ sera la coupure pour la fonction définie par (*).

*M. Tomić* (Belgrade)

763:

**Pogorzelski, W. Problèmes aux limites discontinus dans la théorie des fonctions analytiques.** Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 311–317. (Russian summary, unbound insert)

Let $S$ be the finite union of disjoint domains $S_1, S_2, \cdots, S_k$ in the plane with the boundaries of these domains being arcs, $l_1, l_2, \cdots, l_m$, of certain smoothness. $S_E$ denotes the complement of $S$ minus certain other arcs $L_1, \cdots, L_p$. The Hilbert problem treated by the author consists of finding a function $\Phi(z)$ analytic in each of the domains $\{S_j\}$ and in $S_E$ such that $\Phi^+(z) = G(z)\Phi^-(z) + g(z)$ on each of the arcs. $G$ and $g$ are given functions at least piecewise continuous on the arcs and $\Phi^+$ and $\Phi^-$ denote the limits on approach from the two sides of an arc. There is included a note on Riemann's problem, that is, to find $\Phi(z)$, analytic in $|z| < 1$ with $\operatorname{Re}[(a(\theta) + ib(\theta))\Phi(\exp i\theta)] = f(\theta)$. Again $a$, $b$ and $f$ are to be piecewise continuous.

*R. C. MacCamy* (Pittsburgh, Pa.)

764:

**Alenicyn, Yu. E. An extension of the principle of subordination to multiply connected regions.** Dokl. Akad. Nauk SSSR **126** (1959), 231–234. (Russian)

Let $G$ be a finitely connected domain in the complex plane with no isolated boundary points and not containing the point at infinity. In generalizing the Schwarz lemma, Ahlfors [Duke Math. J. **14** (1947), 1–11; MR **9**, 24] introduced a certain extremal function $F(z, a)$ for the domain $G$, and proved the following lemma. If $f(z)$ is subordinate to $g(\tau)$ with respect to $z_1$ and the domain $G$ and if $f(z)$ is meromorphic in $G$ and regular at $z_1$, then $|f'(z_1)| \leqq |g'(0)| F'(z_1, z_1)$. The author uses this lemma to derive a variety of theorems on the bounds for certain classes of functions meromorphic in $G$. For example, suppose that $f(z)$ is regular and $|f(z)| < M$ in $G$ and $f(a) = 0$. Let $\alpha$ be a simple uncovered point for $f(z)$ (meaning that $\log(f(z) - \alpha)$ is regular in $G$). If $0 < |\alpha| < M$ then

$$|f'(a)| \leqq \frac{2M^2 |\alpha| F'(a, a)}{M^2 - |\alpha|^2} \log \frac{M}{|\alpha|}.$$

*A. W. Goodman* (Lexington, Ky.)

765:

**Ugodčikov, A. G. On trigonometric interpolation of conformal mapping functions.** Ukrain. Mat. Ž. **11** (1959), 111–114. (Russian)

Let $S$ denote a simply-connected region of the finite plane whose frontier consists of a smooth closed Jordan curve $L$. Let $\phi(s)$ denote a continuous determination of the angle from the positive real axis to the ray tangent to $L$ at the point corresponding to the parameter $s$ in some fixed arclength parametrization of $L$. Let $\omega$ denote a univalent conformal map of $|z| < 1$ onto $S$ satisfying $\omega(0) = 0$ and carrying a given fixed direction at 0 into a given direction at 0. Let $\omega_n$ denote a (normalized) polynomial of degree at most $n$ whose real part agrees with the real part of $\omega$ at the points $\exp(\pi j i / n)$, $j = 1, 2, \cdots, 2n$. The following theorem is established: If $\varphi$ satisfies a Lipschitz condition, then the sequence $(\omega_n)$ converges uniformly to $\omega$ in the closed unit disk.

*M. H. Heins* (Urbana, Ill.)

766:

**Osserman, Robert. A lemma on analytic curves.** Pacific J. Math. **9** (1959), 165–167.

Utilisation du revêtement universel d'une surface de Riemann et des fonctions d'automorphie de la projection pour établir le résultat suivant: Toute courbe fermée sur une surface de Riemann est homotope à une courbe fermée analytique et homologue (homologie singulière) à une somme finie de courbes analytiques de Jordan.

*L. Fourès* (Marseille)

767:

**Kusunoki, Yukio. Contributions to Riemann-Roch's theorem.** Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 161–180.

Utilisation de l'espace vectoriel de certaines différentielles Abéliennes, de certaines intégrales Abéliennes, de certaines fonctions méromorphes, et des duals de ces espaces, pour établir le théorème de Riemann-Roch pour des surfaces closes.

La même méthode appliquée au cas d'une surface de Riemann non compacte fournit une extension d'un théorème de Nevanlinna (existence d'une formule du type Riemann-Roch) valable pour des différentielles à carré intégrable, sur des surfaces de Riemann de genre fini et ne possédant pas de fonction de Green (classe $O_G$).

Comme application l'auteur obtient en particulier la représentation injective de toute surface de Riemann $\in O_{HD}-O_G$ dans l'espace produit $\prod_{i=1}^{\infty} C_i$ (où $C_i=C$: plan complexe) ($O_{HD}$: classe des surfaces de Riemann n'admettant pas de différentielle totale harmonique $\neq 0$, à carré intégrable sur $R$; $O_{HD}-O_G\neq\emptyset$ seulement si $R$ est de genre infini).

*L. Fourès* (Marseille)

768:

**Wermer, John. The maximum principle for bounded functions.** Ann. of Math. (2) **69** (1959), 598–604.

Soit $R$ une surface de Riemann. Soit $S\subset R$ où $S$ est limité par une courbe fermée simple $\Gamma$. Soit $\phi$ l'ensemble des fonctions analytiques bornées sur $S\cup\Gamma$. Supposons que $\phi$ vérifie les trois conditions: (1) $\forall f\in\phi$ et $\forall x\in S$, $|f(x)|\leq\sup|f(t)|$ $(t\in\Gamma)$; (2) $\forall x\in S\cup\Gamma$ et $\forall y\in S\cup\Gamma$ avec $x\neq y$, $\exists g\in\phi$ avec $g(x)\neq g(y)$; (3) $\forall p\in\Gamma$, $\exists f\in\phi$ avec $f'(p)\neq 0$. Dans ces conditions $\exists$ une surface de Riemann $R^*$ et $S^*\subset R^*$ où $S^*$ est limité par une courbe simple fermée $\Gamma^*$ telle que (i) $S^*\cup\Gamma^*$ soit compact, (ii) $S^*\subset\psi(S)$ où $\psi$ est une injection analytique, (iii) $f^*=f\circ\psi^{-1}$ définie sur $\psi(S)$ admet une extension analytique sur $S^*\cup\Gamma^*$, et (iv) $\psi$ admet une extension $\bar{\psi}$ biunivoque analytique sur $S\cup\Gamma$ avec $\bar{\psi}(\Gamma)=\Gamma^*$. Ce théorème fondamental a pour conséquences d'intéressantes propositions: (v) $S$ est de genre fini; (vi) $(S^*,\psi)$ est défini à une equivalence conforme près; (vii) le complément de $\psi(S)$ dans $S^*$ est enlevable (pour les fonctions analytiques bornées).

*L. Fourès* (Marseille)

769:

**Valiron, Georges. ★Fonctions entières d'ordre fini et fonctions méromorphes.** D'après un manuscrit revu et préfacé par Henri Milloux. Monographies de "L'Enseignement Mathématique", No. 8. Institut de Mathématiques, Université, Genève, 1960. 150 pp. 20 fr. s.

A reprinting of the lecture series published posthumously in Enseignement Math. (2) **4** (1958), 1–18, 124–177, 229–271; **5** (1959), 1–28 [MR **21** #2048, 5731].

770:

**Edrei, Albert; Fuchs, Wolfgang H. J. On the growth of meromorphic functions with several deficient values.** Trans. Amer. Math. Soc. **93** (1959), 292–328.

Mit den Bezeichnungen der Wertverteilungstheorie sei

$$\lambda(f) = \limsup[T(r,f)/\log r],$$
$$\mu(f) = \liminf[T(r,f)/\log r],$$
$$N(r) = N(r,0)+N(r,\infty),$$
$$\kappa(f) = \limsup[N(r,f)/T(r,f)],$$
$$\delta(\tau) = 1-\limsup[N(r,\tau)/T(r)].$$

Mit Hilfe geschickter Abschätzungen des Weierstrass-schen Primfaktors $E(u,q)$ gewinnen die Verf. Sätze über die Beziehungen zwischen $T$, $\lambda$, $\mu$, $\kappa$, $\delta$ und dem Geschlecht $q$ kanonischer Produkte.

Theorem 1: Sei $f(z)$ meromorph und $\lambda(f)<\infty$. Dann gilt mit richtiger Grössenordnung

$$\kappa(f)\geq|\sin\pi\lambda|(2\cdot 2\lambda+\tfrac{1}{2}|\sin\pi\lambda|)^{-1}.$$

Theorem 2: Seien $\{a_\nu\}$, $a_\nu<0$, die Nullstellen einer ganzen Funktion $f(z)$, wobei $n$-fache Nullstellen $n$ mal in $\{a_\nu\}$ erscheinen; sei das Geschlecht $q$ des zur Folge $\{a_\nu\}$ gehörigen kanonischen Produktes endlich und $\geq 1$. Dann gilt $\delta(0)>A(1+\log q)^{-1}$ mit einer festen Zahl $A$. Die Theoreme 3 und 5 geben Ungleichungen für die Änderung von $T(r)$ bei Änderung des Arguments $r$ und Beziehungen zwischen $\lambda$ und $\mu$, wenn $\kappa$ hinreichend klein {in (2.14) lies $\mu$ anstelle von $u$}. Theorem 4: Es habe die meromorphe Funktion $f(z)$ wenigstens zwei defekte Werte $\alpha$, $\beta$; es sei $\gamma=\max[1-\delta(\alpha),1-\delta(\beta)]$. Dann gilt (i) $\mu\geq 1$ wenn $\gamma=0$; (ii) $\mu\geq -\log[\gamma(2-\gamma)]/\log(1+4/[\gamma(1-\gamma)])$ wenn $\gamma>0$. Theorem 6: Sei $f(z)$ meromorph, $\mu(f)<\infty$, sei $p$ ganzzahlig bestimmt durch $p-\frac{1}{2}\leq\mu<p+\frac{1}{2}$, sei $0<\beta\leq\frac{1}{2}$. Ist dann $\kappa(f)<\beta[5e(p+1)]^{-1}$, so gilt (i) $p\geq 1$, (ii) $|\lambda-p|<\beta/10$, (iii) $p-\beta\leq\mu\leq\lambda<p+\beta/10$. Hieraus folgt: Cor. 1. $\kappa(f)=0$ impliziert $\lambda=\mu$ ganzzahlig endlich oder unendlich; Cor. 2. $\kappa(f)<[10e(\mu+\frac{1}{2})]^{-1}$ impliziert, dass alle Defekte ursprungsunabhängig sind. Theorem 7: Ist $f(z)$ ganz, $\lambda(f)<\infty$ und $\sum\delta(\tau)=2$, so ist $\lambda=\mu$ ganzzahlig. Lemma 1: Ist $f(z)$ ganz von endlicher Ordnung, ist $\sum_{\tau\neq\infty}\delta(\tau,f)>1-\gamma$ und $\delta(\infty,f)>1-\gamma$ mit $0<\gamma<1$, so gilt

(i) $$1-\gamma<\liminf[T(r,f')/T(r,f)]\leq\limsup[T(r,f')/T(r,f)]<1+\gamma,$$

(ii) $\kappa(f')\leq 4\gamma/(1-\gamma^2)$. [Vgl. hierzu auch Wittich, Math. Z. **69** (1958), 237–238; MR **20** #2448.]

*F. Huckemann* (Giessen)

771:

**Remizova, M. P. On regions of values of analytic functions represented by the sum and product of Stieltjes integrals.** Ukrain. Mat. Ž. **11** (1959), 175–182. (Russian. English summary)

Let $f(\zeta)$ be representable in the form

$$(*)\qquad f(\zeta)=\sum_{i=1}^{n}\int_0^b G_i(\zeta,t)\,d\mu_i(t),$$

where each $G_i(\zeta,t)$ is regular in $\zeta$ in a domain $F_i$ and continuous in $t$ for $0\leq t\leq b$, and $\mu_i(t)$ is nondecreasing with $\int_0^b d\mu_i(t)=1$. Let $\zeta$ be a fixed point in $\bigcap F_i$ and let $D$ be the domain of values for $f(\zeta)$ as the $\mu_i(t)$ run independently through the class of admissable functions. Then $D$ is

connected, convex, and closed. Let $\Phi(w)$ be continuous in $D$. The author proves a number of theorems concerning those functions that furnish extreme values for $\Phi(f(\zeta))$ where $f(\zeta)$ has the representation (*). Similar results are obtained when $\sum$ is replaced by $\prod$ in (*). She then applies her results to solve a large variety of extremal problems in the class of functions regular in the ring $q \leq |\zeta| < 1$,

$$f(\zeta) = 1-\alpha-\beta+\frac{\alpha}{\pi}\int_{-\pi}^{\pi}\frac{1+\zeta e^{-it}}{1-\zeta e^{-it}}\,d\mu_1(t) + \frac{\beta}{\pi}\int_{-\pi}^{\pi}\frac{1+a\zeta^{-1}e^{it}}{1-a\zeta^{-1}e^{it}}\,d\mu_2(t).$$

*A. W. Goodman* (Lexington, Ky.)

772:

**Gal'perin, I. M. On the $\arg f'(z)$ of a $p$-leaved star-like function.** Ukrain. Mat. Ž. **11** (1959), 207–210. (Russian)

It is shown that if $f(z)$ is defined for $|z|<1$ by

$$f(z) = z^p \exp\left[-p\pi^{-1}\int_0^{2\pi}\log(1-ze^{-it})d\mu(t)\right],$$

where $\mu$ is non-decreasing on $\{0 \leq t \leq 2\pi\}$ and satisfies $\mu(2\pi)-\mu(0)=2\pi$, then for a given $z$, $|z|<1$, $\arg f'(z)$ is maximized when $\mu$ is a certain step function with a single discontinuity. (Similarly, $\arg f'(z)$ is minimized when $\mu$ is a certain step function with a single discontinuity.)

*M. H. Heins* (Urbana, Ill.)

773:

**Rényi, A. Some remarks on univalent functions.** Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 2, 111–121 (1959). (Bulgarian and Russian summaries)

Modifying the proof of W. Kaplan [Michigan Math. J. **1** (1952), 169–185; MR **14**, 966] pertaining to close-to-convex functions, the author shows that if

$$(1) \qquad |\arg(f'(z)/\phi'(z))| \leq \beta \leq \tfrac{1}{2}\pi, \quad |z|<1,$$

where

$$(2) \qquad f(z) = z+\sum_2^{\infty} a_n z^n$$

is regular in $|z|<1$ and $\phi(z)$ is regular, univalent and convex in $|z|<1$, then

$$(3) \qquad |a_n| \leq 1+\frac{2\beta}{\pi}(n-1) \quad (n=2,3,\cdots).$$

When (1) holds $f(z)$ is said to be close-to-convex of type $\beta$. [The special cases $\beta=\frac{1}{2}\pi$ and $\beta=\frac{1}{3}\pi$ of (3) were obtained earlier by M. O. Reade, C. R. Acad. Sci. Paris **239** (1954), 1758–1759; Ann. Acad. Sci. Fenn. Ser. A. I. no. 215 (1956); MR **16**, 579; **17**, 1069.]

The author uses (3) to improve his earlier set of inequalities on $|a_n|$ when $w=f(z)$ maps $|z|<1$ on domains of boundary rotation $\alpha$, $3\pi \leq \alpha \leq 4\pi$, i.e., when

$$(4) \qquad \int_0^{2\pi}|\mathrm{R}\{1+zf''(z)/f'(z)\}|d(\arg z) \leq \alpha$$

[see A. Rényi, Publ. Math. Debrecen **1** (1949), 18–23; MR **11**, 92]. He shows that $f(z)$ is close-to-convex of type $\beta=\frac{1}{2}(\alpha-2\pi)$, whence it follows from (3) that

$$(5) \qquad |a_n| \leq 1+\frac{(\alpha-2\pi)}{2\pi}(n-1) \quad (n=2,3,\cdots;\ 3\pi \leq \alpha \leq 4\pi).$$

*M. S. Robertson* (New Brunswick, N.J.)

774:

**Rényi, A. Some remarks on univalent functions. II.** Ann. Acad. Sci. Fenn. Ser. A. I. no. 250/29 (1958), 7 pp.

The author generalizes the notion of close-to-convex functions due to W. Kaplan [Michigan Math. J. **1** (1952), 169–185; MR **14**, 966] as follows. Let $f(z)$ be regular and univalent for $|z|<R$, $f(0)=0$, $f'(0)=1$. The function is called close-to-convex of type $\beta$ $(0 \leq \beta \leq \frac{1}{2}\pi)$ in $|z|<R$ if there exists a function $\varphi(z)$ which is regular and univalent in $|z|<R$, maps this circle onto a convex domain and satisfies $|\arg(f'(z)/\varphi'(z))|<\beta$ for $|z|<R$. Various estimates for coefficients of close-to-convex functions of type $\beta$ are announced as well as a sharp distortion theorem. The main result proved in the present paper is the following. Let $r(\beta)$ be the radius of close-to-convexity of type $\beta$, i.e., the largest number such that every function $f(z)$ which is regular and univalent in the unit circle is close-to-convex of type $\beta$ in the circle $|z|<r(\beta)$. Then

$$r(\beta) \geq \operatorname{tgh}(\max(\tfrac{1}{2}\beta, \tfrac{1}{4}(\beta+\log 3))).$$

The determination of the exact value of $r(\beta)$ remains open.

*M. Schiffer* (Stanford, Calif.)

775:

**Vituškin, A. G. Necessary and sufficient conditions a set should satisfy in order that any function continuous on it can be approximated uniformly by analytic or rational functions.** Dokl. Akad. Nauk SSSR **128** (1959), 17–20. (Russian)

In this note the proof is given of necessary conditions for approximation of continuous functions by analytic functions. [For sufficient conditions see the same Dokl. **123** (1958), 778–781, 959–962; MR **21** #2056, #2057.] Let $e$ be a bounded set of points in the $z$-plane, $Ce$ the complement of $e$ with respect to the $z$-plane, $\gamma(e)$ the analytic capacity of $e$ [cf. the previous reviews], and $\sigma(z,r)$ a closed circle in the $z$-plane. Then a necessary and sufficient condition for the uniform approximability of every complex-valued function $f(z)$ defined and uniformly continuous on $e$, by functions analytic in some neighborhood of $e$, is that for every circle $\sigma(z,r)$

$$\gamma(Ce \cap \sigma(z,r)) = r = \gamma(\sigma(z,r)).$$

*M. Tomić* (Belgrade)

776:

**Walsh, J. L. Note on approximation by bounded analytic functions (Problem α).** Math. Z. **72** (1959/60), 47–52.

Let $f(z)$ be analytic on a point set $E$, $p$ a non-negative integer, $n=1, 2, \cdots$, $0<\alpha \leq 1$. A relation is deduced between the approximation to $f(z)$ by functions $f_n(z)$ which are analytic on some set $D$ and that by polynomials $p_n(z)$ of degree $n$. The result is applied to approximation by polynomials. Theorem 1: If $E$ consists of a finite number of mutually disjoint closed Jordan arcs and regions, and $D$ is an open set $\supset E$; if each $f_n(z)$ is analytic on $D$ and (1) $|f(z)-f_n(z)| \leq A_1/n^{p+\alpha}$ on $E$, (2) $|f_n(z)| \leq A_2 R^n$ in $D$; then there exist $p_n(z)$'s such that (3) $|f(z)-p_n(z)| \leq A_3/n^{p+\alpha}$ on $E$.

As the first application, a new proof is given for the known result (theorem 2): If $C_1, C_2, \cdots, C_r$ are mutually exterior analytic Jordan curves, $f(z)$ analytic in the interior of each $C_j$ and continuous in the corresponding closed region $B_j$, then $f(z)$ belongs to $L(p,\alpha)$ $(0<\alpha \leq 1)$ on

$E = B_1 + B_2 + \cdots + B_r$ if and only if there are polynomials satisfying (3), where $L(p, \alpha)$ denotes a certain Lipschitz class.

Finally an analogue to theorem 1 is deduced. It deals with approximation on an analytic Jordan curve, containing the origin in its interior, by polynomials in $z$ and $1/z$. *H. Kober* (Birmingham)

777:

**Tumarkin, G. C. Convergence of sequences of analytic and meromorphic functions.** Dokl. Akad. Nauk SSSR **129** (1959), 280–283. (Russian)

Necessary and sufficient conditions are given for the uniform convergence of a sequence of functions $\{f_k(z)\}$ analytic in $|z| < 1$ and of bounded characteristic

$$\left(\int_0^{2\pi} \ln^+|f_k(re^{i\theta})|d\theta \leqq C, \quad 0 < r < 1, \text{ all } k\right).$$

Conditions are expressed in terms of an auxiliary function

$$\psi_k(t) = \lim_{r\to 1} \int_0^t \ln|f_k(re^{i\theta})|d\theta - \pi \sum_{0<\arg a_{kj}\leq t} (1-|a_{kj}|^2)$$

and its limit $\psi$ when $k \to \infty$; $a_{kj}$ are zeros of the functions $f_k$. The limit function $f(z) = \lim_{k\to\infty} f_k(z)$ can then be represented as

$$e^{i\gamma} b(z) \exp(2\pi)^{-1} \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \psi^*(\theta)d\theta,$$

where $\gamma$ is a real number, $b(z)$ is Blaschke's product with zeros $\alpha_j$ and $\psi^*(\theta)$ is obtained by modifying $\psi(\theta)$. In the cases when the limit function is identically 0 or when the functions $f_k(z)$ admit special representations, the conditions for the uniform convergence are simplified.

For sequences of meromorphic functions similar conditions are derived using auxiliary functions depending on zeros and poles of $f_k(z)$ and replacing the ordinary metric by the spherical one [cf. by the author, same Dokl. **98** (1954), 739–741, 909–912; MR **16**, 686, 687].

*V. Linis* (Ottawa, Ont.)

## SPECIAL FUNCTIONS

See also 689, B1105, B1130.

778:

**Mangeron, D. New classes of functions related to the equations with "total derivatives".** Bul. Inst. Politehn. Iaşi (N.S.) **4** (8) (1958), no. 3–4, 73–74. (Russian and Romanian summaries)

The author announces his intention to apply a theory of boundary-value problems, developed by himself and others in recent papers, to generalized forms of the Bessel and Legendre differential equations.

*F. W. J. Olver* (Teddington)

779:

**Robin, Louis. ★Fonctions sphériques de Legendre et fonctions sphéroïdales. Tome III.** Collection Technique et Scientifique du C. N. E. T. Gauthier-Villars, Paris, 1959. viii + 289 pp. Broché: 5.500 francs; Cartonné: 6.000 francs.

Le tome III de cet ouvrage s'ouvre sur les formules et théorèmes d'addition relatifs aux fonctions de Legendre, généralement établis ici conformément aux méthodes et aux notations de M. René Lagrange. En ce qui concerne la formule:

$$P_n[bb' + \sqrt{((b^2-1)(b'^2-1))}\,\mathrm{ch}\,\zeta] = P_n(b)P_n(b') + 2\sum_{m=1}^{\infty} P_n^m(b)P_n^{-m}(b')\,\mathrm{ch}\,m\zeta,$$

des précisions supplémentaires sont données sur les conditions de convergence du développement. Deux rectifications importantes sont signalées: la première relative à la valeur du développement précédent lorsque le segment $\mu_1\mu$ rencontre la demi-droite $(-\infty, -1)$ du plan $\mu$, cas où la formule n'est plus valable, son 1er membre devant être alors remplacé par $P_n(\mu) \mp 2i \sin n\pi P_n(-\mu)$ ($-$ si $\Im(\mu) > 0$, et $+$ dans le cas contraire) [$\mu = bb' + \sqrt{((b^2-1) \times (b'^2-1))}\,\mathrm{ch}\,\zeta$ et $\mu_1 = bb' + \sqrt{((b^2-1)(b'^2-1))}$]; la seconde précise qu'il est impossible d'étendre la formule au cas où ne serait pas satisfaite la double inégalité:

$$\left|\frac{b'-1}{b'+1}\right| < \left|\frac{b+1}{b-1}\right| < \left|\frac{b'+1}{b'-1}\right|.$$

Un résumé de l'important travail de P. Henrici (1955) relatif à cette question est donné.

Un chapitre est ensuite consacré aux zéros ou racines des fonctions $P_n^m(\mu)$ et $Q_n^m(\mu)$. Après une discussion des différents cas où $m$ et $n$ sont réels, puis où $n$ est de la forme $-\frac{1}{2} + ip$, on y trouve l'étude détaillée des diverses équations où l'inconnue est le degré des fonctions de Legendre, avec l'exposé du travail de Macdonald relatif au nombre de racines complexes de l'équation: $P_n^m(\cos\alpha) = 0$, où $m$ et $\alpha$ sont donnés réels. L'auteur signale à ce propos une erreur de Macdonald concernant la réalité de certaines racines.

Le chapitre suivant est consacré à la séparation des variables dans l'équation $\Delta U = 0$, quand le système de coordonnées n'est pas sphérique. La recherche des solutions fondamentales de l'équation de Laplace conduit à la résolution des problèmes de Dirichlet intérieur et extérieur pour les ellipsoïdes de révolution. Le développement de l'inverse de la distance de 2 points en séries de solutions fondamentales dans le cas de l'ellipsoïde de révolution aplati conduit l'auteur à donner une démonstration de la convergence du développement notablement plus simple que celle donnée antérieurement par Heine et Hobson. Les mêmes problèmes sont traités avec le découpage de l'espace par des tores.

L'étude détaillée des fonctions toriques, $P^m_{n-1/2}$ et $Q^m_{n-1/2}$, est présentée ensuite. L'auteur donne notamment la théorie de D. P. Banerjee du développement d'une fonction en série de telles fonctions, théorie qu'il complète et rectifie sur divers points. Beaucoup d'exemples sont donnés à cette occasion (avec certaines rectifications de résultats publiés antérieurement). On insiste un moment sur l'exemple très intéressant de l'intégrale de Fourier-Legendre où la variable d'intégration est le degré de la fonction de Legendre (théorie de T. MacRobert) et où les théorèmes relatifs aux développements considérés ont la forme des propositions d'équiconvergence. Une étude longue et détaillée (pp. 137–182) des fonctions coniques de Mehler termine le chapitre. Après une introduction aux propriétés des fonctions de la calotte sphérique, les mêmes problèmes sont traités, en particulier le développement de

l'inverse de la distance de 2 points en une intégrale de solutions fondamentales suivi de la théorie de Banerjee sur les séries de fonctions de Mehler avec de nombreux exemples, empruntés à Mehler, Heine, Fock. Les formules d'addition sont signalées, les développements asymptotiques, et là encore, l'auteur signale des erreurs concernant des publications antérieures.

Une place importante est donnée à la représentation d'une fonction quelconque de 2 variables par une intégrale triple contenant la fonction $K_p$ (avec cas particulier pour les fonctions d'une seule variable et l'intégrale de Fourier-Mehler). On termine par une application de ces fonctions de Mehler à la résolution de problèmes de Dirichlet (équations de Laplace et Helmholtz, demi-cône infini, sphères ne se coupant pas, tore coupant son axe, lentille sphérique).

Vient ensuite un chapitre sur "les fonctions qui se rattachent étroitement à celles de Legendre"; polynômes et fonctions de Gegenbauer et fonctions sphéroïdales. On établit la relation bijective existant entre l'ensemble des polynômes de Gegenbauer et celui des fonctions associées de Legendre de première espèce telles que la somme de leur degré et de leur ordre soit un entier $\geq 0$. L'auteur définit alors une nouvelle fonction de Gegenbauer de seconde espèce liée à la fonction associée de Legendre correspondante par la même relation.

Ainsi, les fonctions de Gegenbauer des 2 espèces vérifient les mêmes relations de récurrence, contrairement à ce qu'on en ait dit sur la fonction de seconde espèce classique introduite par Gegenbauer et adoptée depuis. Dans le corps de cette étude est incluse l'étude des propriétés des harmoniques sphériques à plus de deux dimensions.

C'est l'équation sphéroïdale, généralisation de l'équation de Mathieu, qui est ensuite envisagée sous de multiples formes, ainsi que les fonctions sphéroïdales, fonctions propres de cette équation, dont on donne le développement en séries de fonctions associées de Legendre et de fonctions de Bessel sphériques. Une nomenclature des diverses notations utilisées pour ces fonctions et de la correspondance entre ces notations est très utilement offerte au lecteur, ainsi que la liste des tables numériques existantes.

Enfin, en annexe, on trouve la liste et le contenu des tables numériques des fonctions de Legendre parues avant la fin de 1958. Cette liste est divisée en 10 parties, d'après le contenu des tables, pour faciliter sa consultation.

La division très nette des chapitres, l'ordre adopté, la précision apportée à l'introduction et à l'utilisation des notations rend aisée et agréable la lecture de ce livre qui apparaît désormais comme l'outil indispensable à tout usager des fonctions sphériques. *R. Campbell* (Caen)

780:

**Karmazina, L. N. On the asymptotics of spheroidal wave functions.** Vyčisl. Mat. **5** (1959), 72–78.

The functions studied are the solutions of the differential equation

$$\frac{d}{dz}((1-z^2)y'(z)) + \left(\Lambda + \gamma^2(1-z^2) - \frac{\mu^2}{1-z^2}\right)y(z) = 0,$$

where $\Lambda$ and $\gamma$ are parameters, and it is desired to find asymptotic solutions as $\Lambda \to \infty$, $\gamma$ and $z$ fixed, $-1 < z < 1$.

The method followed, in outline, is as follows. the changes of variables $z = \cos x$, $w(x) = (\sin x)^{1/2} y(\cos x)$ change the differential equation to

$$w''(x) + (\lambda^2 + \gamma^2 \sin^2 x - ((4\mu^2 - 1)/4 \sin^2 x))w(x) = 0,$$

where $\lambda^2 = \Lambda + \frac{1}{4}$. An integral equation for $w$ is found in terms of the solutions of the differential equation $v''(x) + (\lambda^2 - ((4\mu^2 - 1)/4x^2))v(x) = 0$, which can be completely solved in terms of Bessel functions. Iteration of the integral equation leads to a series which is shown to be an asymptotic series for $w$.

As a typical result, the author shows that if $m$ is fixed, $m = 0, 1, 2, \cdots$, $\varepsilon \leq x \leq \pi - \varepsilon$, then

$$P_\nu^m(\cos x) = (-1)^m(\nu + \tfrac{1}{2})^m (x/\sin x)^{1/2}\{J_m((\nu + \tfrac{1}{2})x) - ((4m^2 - 1)/(8\nu + 4))(\cot x - x^{-1}) Y_m((\nu + \tfrac{1}{2})x) + O(\nu^{-5/2})\},$$

where $P_\nu^m(z)$ is the associated Legendre function.

*P. G. Rooney* (Toronto)

781:

**Olver, F. W. J. Uniform asymptotic expansions for Weber parabolic cylinder functions of large orders.** J. Res. Nat. Bur. Standards Sect. B **63B** (1959), 131–169.

The author's researches improve upon the earlier work of Erdélyi, Kennedy and McGregor [J. Rational Mech. Anal. **3** (1954), 459–485; MR **16**, 33] and others in three important ways. First, expansions are found; that is, complete asymptotic series are derived. Second, the number of representations needed to describe a standard Weber function is considerably reduced. Third, the results are uniform with respect to arg $\nu$, where $\nu$ is the order. The expansions obtained by the reviewer [Arch. Rational Mech. Anal. **2** (1958), 129–150; MR **20** #5919] are also uniform in arg $\nu$ but are merely extensions of the Erdélyi-Kennedy-McGregor representations.

The expansions are obtained by application of Langer's method for the study of asymptotic solutions of differential equations with turning points. The particular theorems used are those derived by the author [Philos. Trans. Roy. Soc. London. Ser. A **250** (1958), 479–517; MR **20** #1012].

Numerical examples are given: evaluation of $W(a, \pm x)$ and $W'(a, \pm x)$ for $a = 8$ and $x = 7$; evaluation of the Hermite polynomial $H_n(x)$ for $n = 10$ and $x = 4$. Expansions for the real zeros of various Weber functions are found and sample computations presented.

This article is the definitive one on the asymptotic behavior of Weber functions of large order.

*N. D. Kazarinoff* (Moscow)

782:

**Erber, Thomas. Inequalities for hypergeometric functions.** Arch. Rational Mech. Anal. **4**, 341–351 (1960).

This important paper discusses the general problem of deriving families of inequalities for the Gauss hypergeometric function $F(\alpha, \beta; \gamma; z)$. These inequalities are potentially of great usefulness in general in numerical analysis, and in particular in the theory of asymptotic approximations to these functions. The author outlines the known inequalities for the Gamma function, and then gives two elementary inequalities for the function expressed as an infinite series and for the function expressed as an Euler type integral. These are:

$$|F| \leq \cos \tfrac{1}{2}\theta \, \frac{\Gamma(|\gamma|)\Gamma(|\gamma| - |\alpha| - |\beta|)}{\Gamma(|\gamma| - |\alpha|)\Gamma(|\gamma| - |\beta|)},$$

if $|\gamma| > |\alpha| + |\beta|$; and when $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$, if $\operatorname{Re}\alpha < 0$,

$$|F| < \frac{B(\operatorname{Re}\beta,\ \operatorname{Re}(\gamma-\beta))}{|B(\beta,\gamma-\beta)|}(1+|z|)^{\operatorname{Re}(-\alpha)}\Phi(\alpha,z),$$

but if $\operatorname{Re}\alpha > 0$,

$$|F| < \frac{B(\operatorname{Re}\beta,\ \operatorname{Re}(\gamma-\beta))}{|B(\beta,\gamma-\beta)|}\cos[\tfrac{1}{2}\arg(-z)]^{\operatorname{Re}(-\alpha)}\Phi(\alpha,z),$$

where

$$\Phi(\alpha,z) = \exp\{\tfrac{1}{2}[1-\operatorname{sgn}(\operatorname{Im}\alpha\ \operatorname{Im}z)][\operatorname{Im}(\alpha)\arg(1-z)]\}$$

and $B(\alpha,\beta)$ is a Beta function.

He then goes on to consider the Mellin-Barnes integral representation of the function, and deduces the further general inequality

$$|F| < \frac{1}{2\pi}\left|\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\right||z|^{-\gamma/2}\exp\{-\operatorname{Im}(\alpha)\arg(-z)\}$$

$$\times\int_{-\infty}^{\infty}|\Gamma([\operatorname{Re}\alpha-\tfrac{1}{2}\gamma]+iv)\Gamma([\operatorname{Re}\beta-\tfrac{1}{2}\gamma]+i[v+\operatorname{Im}(\beta-\alpha)])|\times e^{-v\arg(-z)}dv.$$

The paper includes with an inequality for Appell's function $F_2$, and a long list of references to similar works, which even so is far from complete.

*L. J. Slater* (Cambridge, England)

783:

**Sretenskiĭ, L. N. On Čaplygin functions.** Prikl. Mat. Meh. **23** (1959), 574–575 (Russian); translated as J. Appl. Math. Mech. **23**, 807–810.

The Chaplygin functions in question are the hypergeometric series $F(a_n, b_n; n+1; z)$ in which $a_n$ and $b_n$ are the two roots of the quadratic equation

$$t^2-(n-\beta)t-\tfrac{1}{2}\beta n(n+1) = 0.$$

The author takes $\beta = 5/2$ and determines those cases in which the Chaplygin functions can be expressed as a finite combination of elementary functions. This, it turns out, happens if $s = (24n^2+25)^{1/2}$ is an integer. Accordingly, the author discusses the diophantine equation $s^2-24n^2 = 25$ and tabulates some of its solutions together with the corresponding values of $a_n$ and $b_n$.

*A. Erdélyi* (Pasadena, Calif.)

784:

**MacRobert, T. M. Multiplication formulae for the $E$-functions regarded as functions of their parameters.** Pacific J. Math. **9** (1959), 759–761.

Multiplication formulas of MacRobert's $E$-function with respect to the parameters are given. The expressions $E(p; m\alpha_r : q; m\rho_s : ze^{\pm 2\pi i})$ and $\sum_{t,-t} i^{-1}E(p; m\alpha_r : q; m\rho_s : ze^{\pi i})$ ($m$ a positive integer) are represented in terms of other $E$-functions.

*F. Oberhettinger* (Corvallis, Ore.)

785:

**Gillis, Joseph; Weiss, George. Products of Laguerre polynomials.** Math. Comput. **14** (1960), 60–63.

Let $L_n(x)$ be the Laguerre polynomial of degree $n$, and let $C_{rst}$ be the coefficient of $L_t(x)$ in the expansion of $L_r(x)L_s(x)$ as a series of Laguerre polynomials. The authors first give a short proof of Watson's [J. London Math. Soc. **13** (1938), 29–32] closed expression for $C_{rst}$, and then deduce a recurrence relation for fixed $r$ and $s$ and variable $t$. This recurrence relation, they point out, is much more suitable for numerical computation than the closed formula, and more suitable for machine computation than other recurrence relations involving variable $r$ and/or $s$ as well as variable $t$.

*A. Erdélyi* (Pasadena, Calif.)

786:

**Weisner, Louis. Generating functions for Hermite functions.** Canad. J. Math. **11** (1959), 141–147.

Let $L$ be the partial differential operator

$$L = \frac{\partial^2}{\partial x^2}-2x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}$$

Then $Lu = 0$ for $u = y^n v(x)$ if and only if $v(x)$ satisfies

$$\frac{d^2v}{dx^2}-2x\frac{\partial v}{dx}+2nv = 0$$

(Hermite's differential equation). The solutions of this equation are Hermite's functions $v_1 = He_n(x)$ and $v_2 = e^{x^2}He_{-n-1}(ix)$. Here $He_n(x)$ is the $n$th Hermite Polynomial. Thus if $u = u(x, y)$ is expressible as a series of powers of $y$, the coefficients of $y^n$ must be a solution of Hermite's differential equation when $Lu = 0$ is required. The equation $Lu = 0$ admits a 5-parameter group of continuous transformations. This group is used to obtain solutions of $Lu = 0$ and thus generating functions for the Hermite functions. A number of examples are given.

*F. Oberhettinger* (Corvallis, Ore.)

787:

**Weisner, Louis. Generating functions for Bessel functions.** Canad. J. Math. **11** (1959), 148–155.

Let $L$ be the partial differential operator

$$L = x^2\frac{\partial^2}{\partial x^2}+x\frac{\partial}{\partial x}-y^2\frac{\partial^2}{\partial y^2}-y\frac{\partial}{\partial y}+x^2.$$

This operator annuls $u = u(x, y) = v(x)y^n$ if and only if $v(x)$ satisfies

$$x^2\frac{d^2v}{dx^2}+x\frac{dv}{dx}+(x^2-n^2)v = 0$$

(Bessel's differential equation) and hence is a cylinder function of order $n$. Thus every generating function of a set of cylinder functions is a solution of $Lu = 0$. The same considerations as employed elsewhere [see preceding review] are used for the determination of generating functions for Bessel functions. A number of examples are given.

*F. Oberhettinger* (Corvallis, Ore.)

788:

**Carlitz, Leonard. Some $q$-polynomials in two variables.** Math. Nachr. **17**, 224–238 (1959).

The polynomials

$$H_m(x, q) = \sum_{r=0}^{m}\begin{bmatrix} m \\ r \end{bmatrix}x^r,$$

$$G_m(x, q) = \sum_{r=0}^{m}\begin{bmatrix} m \\ r \end{bmatrix}q^{r(r-m)}x^r,$$

where

$$\begin{bmatrix} m \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} m \\ r \end{bmatrix} = \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)}$$

have previously been studied by a number of authors, e.g., L. Carlitz [Ann. Mat. Pura Appl. (4) **41** (1956), 359–373; MR **17**, 1205]. These polynomials satisfy certain orthogonality relations and have various formal properties analogous to those of the ordinary Hermite polynomials. In the present paper the related polynomials

$$H_{m,n}(x, y, q) = \sum_{r=0}^{m} \sum_{s=0}^{n} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{rs} x^r y^s,$$

$$G_{m,n}(x, y, q) = \sum_{r=0}^{m} \sum_{s=0}^{n} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{r(r-m)+s(s-n)-rs} x^r y^s$$

are discussed. §§ 1, 2 contain some preliminaries. In § 3 generating functions for $H_{m,n}(x, y)$ and $G_{m,n}(x, y)$ are obtained; these suggest the introduction of certain additional polynomials $h_{m,n}(x, y)$ and $g_{m,n}(x, y)$, which have properties analogous to those of the former polynomials. In § 4 relations between $H_{m,n}(x, y)$ and $G_{m,n}(x, y)$ and the corresponding polynomials in a single variable are derived; similar results are also obtained for $h_{m,n}(x, y)$ and $g_{m,n}(x, y)$. § 5 contains various operational formulas including recursion formulas. § 6 contains a number of integral formulas involving certain weight functions. Finally in § 7 the question of the biorthogonality of the several polynomials is discussed briefly.

*A. L. Whiteman* (Princeton, N.J.)

## ORDINARY DIFFERENTIAL EQUATIONS

See also 904, B1128.

789:

**Bandić, Ivan. Sur une classe d'équations différentielles non-linéaires du deuxième ordre.** Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II **14** (1959), 143–152. (Serbo-Croatian summary)

L'A. determina alcuni casi nei quali è possibile ridurre ad un'equazione del 1° ordine una equazione differenziale ordinaria del 2° ordine della forma $u(x, y, y', y'') = v(x, y, y', y')$ dove $u$, $v$ sono funzioni omogenee rispetto alle $y, y', y''$ dei gradi $m$ ed $n$ rispettivamente. La riduzione è effettuata facendo uso dell'operatore $\Delta_p(y) = y^{(p)}/y$, detto "derivata relativa di ordine $p$". Nel caso $n=1$, $v \equiv 0$ la riduzione si effettua mediante la posizione $\Delta_1(y) = z$, com'è ben noto. Applicazioni a vari casi particolari.

*R. Conti* (Florence)

790:

**Bajcsay, P. Anwendung der Matrizenrechnung zur Lösung gewöhnlicher, linearer Differentialgleichungssysteme mit variablen Koeffizienten.** Period. Polytech. Elec. Engrg. **3** (1959), 217–231.

When the system is written in matrix form, $x' = Ax + f$ (the prime denoting differentiation), it is assumed that there exists a nonsingular matrix $H$ such that $HAH^{-1} = D$ is diagonal. Then with $y = Hx$, the system reduces to $y' = Dy + By + g$, and an obvious iteration is applied and discussed.

*A. S. Householder* (Oak Ridge, Tenn.)

791:

**Reid, William T. Solutions of a Riccati matrix differential equation as functions of initial values.** J. Math. Mech. **8** (1959), 221–230.

Generalizing a scalar system of differential equations introduced by R. Redheffer [J. Rational Mech. Anal. **5** (1956), 835–848; Math. Ann. **133** (1957), 235–250; MR **19**, 558, 655], the author studies the matrix system

$$\begin{aligned} U_y &= A + DU + UB + UCU, & U(x, y) &= 0 \text{ for } y = x, \\ R_y &= R(B + CU), & R(x, y) &= 1 \text{ for } y = x, \\ S_y &= (D + UC)S, & S(x, y) &= 1 \text{ for } y = x, \\ W_y &= RCS, & W(x, y) &= 0 \text{ for } y = x. \end{aligned} \tag{1}$$

No special restrictions on the coefficient matrices are assumed, but the region $G$ in which these equations hold is convex-like. A clever change of variables reduces this system to a linear system, and using this fact the author is able to show uniqueness as well as that $U$, $R$, $S$, and $W$ are continuous in $x$ and $y$ from the assumed existence of partial derivatives in the $y$-direction. For an arbitrary constant matrix $K$, he shows that

$$U(x, y; K) = U + SK(1 - WK)^{-1}R \tag{2}$$

satisfies the first equation in (1) with initial conditions $U(x, x; K) = K$. The matrix functions $U$, $R$, $S$, and $W$ are said to have property $P$ if

$$\|U(x, y; k)\| < 1 \quad \text{whenever } \|K\| < 1 \tag{3}$$

for all $x, y$ in $G$. The author shows that the condition

$$\begin{pmatrix} D^* + D & A + C^* \\ A^* + C & B + B^* \end{pmatrix} \leqq 0 \tag{4}$$

is sufficient for property $P$ and that it is both necessary and sufficient when $A = A'$, $D = B'$, and $C = C'$; here $A^*$ denotes the adjoint of $A$ and $A'$ the transpose of $A$.

*R. S. Phillips* (Los Angeles, Calif.)

792:

**Redheffer, Raymond. Inequalities for a matrix Riccati equation.** J. Math. Mech. **8** (1959), 349–367.

In the present paper, the author extends the above reviewed work of W. T. Reid on a matrix Riccati equation. Making use of Reid's uniqueness theorem, the author establishes the functional equation

$$U(x, z; K) = U(y, z; U(x, y; K))$$

from which several of Reid's relations result in a very natural way. The major part of the paper is concerned with Reid's property $P$ for which the author obtains a necessary and sufficient condition in terms of the coefficients $A, B, C, D$. He also shows that

$$\left\| \begin{pmatrix} S & U \\ W & R \end{pmatrix} \right\| \leqq 1 \tag{5}$$

is both necessary and sufficient for (4) in the preceding review. For constant matrices with $U = U'$, $R = S'$, $W = W'$, and $R$ non-singular, the purely algebraic equivalence of (2) and (5) is established. Also considered are algebraic conditions under which $U + SK(1 - WK)^{-1}R$ is unitary for all unitary $K$.

*R. S. Phillips* (Los Angeles, Calif.)

793:

**McCarthy, P. J. On the disconjugacy of second-order linear differential equations.** Amer. Math. Monthly **66** (1959), 892–894.

Let $r(x)>0$ and $p(x)$ be real-valued continuous functions on $a\leq x\leq b$. The differential equation

$$(r(x)y')'+p(x)y=0$$

is called disconjugate on $a<x<b$ if no non-trivial solution of it vanishes more than once in that interval. Following an idea of J. H. Barrett [Proc. Amer. Math. Soc. **6** (1955), 247–251; MR **17**, 37] the author introduces a new unknown function $\theta(x)$ by the transformation $y(x)=\rho(x)\sin\theta(x)$, $y'(x)=w(x)r(x)^{-1}\rho(x)\cos\theta(x)$, where $\rho=rw^{-1}(w^2r^{-2}y^2+y'^2)^{1/2}$ and $w(x)$ is an arbitrary positive continuously differentiable function in $a\leq x\leq b$. The resulting differential equation for $\theta(x)$ is of the first order. Its analysis leads to the condition

$$\left|\frac{w}{r}+\frac{p}{w}\right|+\left|\frac{w}{r}-\frac{p}{w}\right|+\left|\frac{w'}{w}\right|\leq\frac{2\pi}{b-a}$$

for disconjugacy. This condition is discussed for various choices of $w(x)$, and some generalizations are proved.

*W. Wasow* (Madison, Wis.)

794:

**Datzeff, A. B. Über die Eigenwerte einiger Differentialgleichungen. II.** C. R. Acad. Bulgare Sci. **12** (1959), 285–288. (Russian summary)

[For part I see same C. R. **12** (1959), 113–116; MR **21** #5765.] Es handelt sich um die Differentialgleichung $y''+f(x)y=0$, $f(x)=f(x,\mu)=\varphi(x)+\mu\Phi(x)>0$. $\mu$ ist ein Parameter, $\varphi(x)>0$ und $\Phi(x)>0$ bezeichnen beschränkte integrierbare Funktionen. Die bekannte Lösung des Cauchyschen Problems für das Intervall $(x_0, x_l)=(0, l)$ läßt sich auf das Intervall $x_0=-\infty$, $x_l=\infty$ ausdehnen. Mit $\Phi(x)=A=2m/\hbar^2$, $\varphi(x)=-AU(x)$, $\mu=E$ erhält man die zeitunabhängige Schrödinger Gleichung für den eindimensionalen Fall. Ist das asymptotische Verhalten von $U$ durch die Bedingung $U(x)\to\infty$ für $x\to\pm\infty$ bestimmt, so ergibt sich für die Differentialgleichung ein diskretes Spektrum. Der Verfasser behandelt sowohl den Fall gerader Funktionen $U(x)=U(-x)$ wie auch den Fall unsymmetrischer Funktionen. Im ersten Fall sind zwei Randwertprobleme zu lösen. In beiden Fällen werden Näherungswerte der Eigenwerte erhalten wie bei einem Intervall $(x_0, x_l)$ bzw. $(0, l)$. Der Grenzübergang $l\to\infty$ führt zu keinerlei Schwierigkeiten grundsätzlicher Art. Auch die Radialgleichung von Schrödinger (bei zylindrischer oder sphärischer Symmetrie) läßt sich so im Intervall $(0, \infty)$ behandeln.

*M. Pinl* (Cologne)

795:

**Butler, John B., Jr. Perturbation of the continuous spectrum of even order differential operators.** Canad. J. Math. **12** (1960), 309–323.

This paper is concerned with the dependence of the resolution of the identity of a differential operator of even order on a parameter. If $L_0$ denotes such a differential operator with a continuous spectrum the perturbed operator is of the form $L_\varepsilon=L_0+\varepsilon q(x)$. The author gives sufficient conditions on $L_0$ and integral inequalities on $q(x)$ which guarantee that the resolution of the identity $E_\varepsilon(\Delta)$ is a regular analytic operator (in the sense of F. Rellich) if $\Delta$ is an interval in the continuous spectrum. There is one application to the operator $L_0=d^4/dt^4$ in $0\leq t<\infty$ with the boundary condition $u(0)=u''(0)=0$. The method generalizes a proof of the reviewer. It is interesting that in the proof one uses the Bessel inequality applied to the operator $L_0$ under other boundary conditions than the given ones.

*J. Moser* (Cambridge, Mass.)

796:

**Reissig, Rolf. Kriterien für die Zugehörigkeit dynamischer Systeme zur Klasse *D*.** Math. Nachr. **20** (1959), 67–72.

Un sistema di due equazioni differenziali

$$x'=f(x,y,t),\quad y'=g(x,y,t) \tag{1}$$

è detto da N. Levinson [Ann. of Math. (2) **45** (1944), 723–737; MR **6**, 173] di classe $D$ se al sistema stesso corrisponde una costante $A$ in modo che per ogni soluzione $x=x(t)$, $y=y(t)$ esiste un $t_0$ tale che

$$x^2(t)+y^2(t)\leq A,\quad t_0\leq t. \tag{2}$$

Tali sistemi godono di importanti proprietà ed ha perciò interesse stabilire, se un sistema (1) appartiene o no alla classe $D$. Effettuando una opportuna trasformazione bicontinua che muta i punti improprii di ogni piano $t=$ cost. nel punto $x=y=0$ e limitando il proprio studio al caso che $f$ e $g$ siano periodiche in $t$, l'A. deduce un criterio di appartenenza ed uno di non appartenenza alla classe $D$ per il sistema (1) utilizzando note condizioni per la instabilità totale e la stabilità, rispettivamente, della soluzione nulla del sistema trasformato.

*R. Conti* (Florence)

797:

**Olech, C. Periodic solutions of a system of two ordinary differential equations.** Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 137–140. (Russian summary, unbound insert)

Let $\dot{x}=g(t,x,y)$, $\dot{y}=h(t,x,y)$, $g$, $h$ continuous, periodic in $t$ of period 1 and non-decreasing in $y$, $x$ respectively, when the other variables are kept constant; if uniqueness conditions are satisfied and a solution exists which is bounded in the future, then a periodic solution of period 1 exists. The particular case $g=y$ was considered by Z. Opial [same Bull. **7** (1959), 71–75; MR **21** #4275].

*J. L. Massera* (Montevideo)

798:

**Mitropol'skiĭ, Yu. A. Periodic solutions of simultaneous non-linear differential equations with non-differentiable right-hand members.** Dokl. Akad. Nauk SSSR **128** (1959), 1118–1121. (Russian)

A system of equations is considered: (1) $dx/dt=\varepsilon X(t,x)$, where $x$ is an $n$-vector, $\varepsilon$ is a small positive parameter, and $X$ is $2\pi$-periodic in $t$, is bounded, and satisfies a Lipschitz-type condition. $\bar{X}(\xi)$ is defined as $(2\pi)^{-1}\int_0^{2\pi}X(t,\xi)dt$ and is assumed to have bounded derivatives to third order at least. The equation $d\xi/dt=\bar{X}(\xi)$ is assumed to have an isolated stationary solution $\xi=\xi_0$, for which $\bar{X}(\xi_0)=0$, $\|d\bar{X}(\xi)/d\xi\|_{t=t_0}\neq 0$.

The author proves that for given $\sigma$ there is a neighbourhood of $\xi_0$ in which there is a solution of (1) $2\pi$-periodic in $t$ such that $|x(t)-\xi_0|<\sigma$, and also that if the Lipschitz condition is strengthened the solution is unique.

*F. M. Arscott* (London)

799:

**Opial, Z. Sur les solutions presque-périodiques des équations différentielles du premier et du second ordre.** Ann. Polon. Math. **7** (1959), 51–61.

En se servant de la méthode de Favard [Acta Math. **51** (1928), 31–81] et en généralisant les théorèmes de Demidovič [Uspehi Mat. Nauk **8** (1953), no. 6 (58), 103–106; MR **15**, 872] et de Gheorghiu [An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **1** (1955), 17–20; MR **18**, 899], l'auteur démontre le théorème suivant.

Si la fonction $f(t, x)$, définie et continue dans le plan $(t, x)$, est une fonction presque périodique de $t$, uniformément par rapport à $x$, et monotone au sens large par rapport à $x$, alors toute solution bornée de l'équation $dx/dt = f(t, x)$ est presque-périodique.

Par la même méthode, l'auteur démontre un théorème analogue pour l'équation du second ordre $d^2x/dt^2 = f(t, x, dx/dt)$, qui généralise un théorème de Corduneanu [Com. Acad. R. P. Romîne **5** (1955), 793–797; MR **17**, 39].

*D. S. Mitrinović* (Belgrade)

800:

**Reghiş, M. Sur la stabilité "d'après la première approximation".** Lucrăr. Şti. Inst. Ped. Timişoara. Mat.-Fiz. **1958**, 135–144 (1959). (Romanian. French and Russian summaries)

Let $\dot{x} = P(t)x$, $x$ being an $n$-vector, $P$ a continuous bounded matrix; let $U(t, t_0)$ be the matrix solution reducing to the identity for $t = t_0$, $x_s(t, t_0)$ the vector solutions which form $U$. Theorems: 1. A necessary and sufficient condition that $\|x_s(t, t_0)\| < A(\gamma)C(t_0)e^{-\gamma(t-t_0)}$, $s = 1, \cdots, n$, be satisfied for any $t \geqq t_0 \geqq 0$, $\gamma \in [0, \alpha)$, where $\alpha$ is fixed and $A(\gamma)$, $C(t)$ are continuous, is that for each $\delta \in [0, \alpha)$ a Lyapunov function $V(x, t)$ exists such that

$$Ne^{\delta t}\|x\| < V < M(\delta)C(t)e^{\delta t}\|x\|, \quad V' = -e^{\delta t}\|x\|.$$

2. Let $m > 1/2$, $\beta < \alpha(2m-1)$, and assume that the property in theorem 1 is satisfied with $C(t) = Be^{\beta t}$; then for each vector function $X(t, x)$ satisfying $\|X\| \leqq L\|x\|^{2m}$, each solution $x(t)$ of $\dot{x} = P(t)x + X(t, x)$ with $\|x(t_0)\|$ small enough satisfies $\|x(t)\| < A(\gamma)e^{-\gamma t}$ for each $\gamma \in [0, \alpha)$.

*J. L. Massera* (Montevideo)

801:

**Mikolajska, Z. Sur la stabilité asymptotique des solutions d'un système d'équations différentielles.** Ann. Polon. Math. **7** (1959), 13–19.

The author considers the system (1) $dx/dt = F(t, x)$, where $x$ is an $n$-vector, $n > 1$, $F(t, x)$ is continuous for all $t$ and $x$, and the solutions of (1) are unique. Let $Z$ be a compact connected subset of the Euclidean $n$-space $E$ such that $E - Z$ consists of disjoint (open) components $\gamma_\iota$. Let $G$ be the union of the $\gamma_\iota$ which are bounded. The author proves the following theorem. Theorem T: If every solution of (1), $\varphi(t, p, 0)$, $p \in Z$, is asymptotically stable, then every solution $\varphi(t, q, 0)$, $q \in G$, is equi-asymptotically stable in the sense of Lyapunov—i.e., there is a neighborhood $\omega$ of $q$ such that $\|\varphi(t, q, 0) - \varphi(t, q', 0)\| \to 0$ as $t \to \infty$ uniformly in $q' \in \omega$. The proof is elementary in the sense that nothing more advanced than the Heine-Borel-Lebesgue theorem is used.

Krasovskiĭ [Prikl. Mat. Meh. **21** (1957), 309–319; MR **19**, 652] proved that if $\varphi(t)$ is a uniformly stable solution of (1) for which $\varphi(t, q, 0)$ is asymptotically stable for all $q$ belonging to some deleted neighborhood $\Omega$ of $\varphi(0)$, then $\varphi(t)$ itself is asymptotically stable and its region of asymptotic stability includes $\Omega$. The author points out that her theorem includes Krasovskiĭ's (for $n > 1$) as a special case and that no assumption of stability for $\varphi(t)$ is necessary. An example is given to show why the assumption of the connectedness of $Z$ is essential—and thus why $n$ must be greater than 1.

*C. S. Coleman* (Claremont, Calif.)

802:

**Nemyckiĭ, V. V. On certain methods of qualitative investigation "in the large" of many-dimensional autonomous systems.** Trudy Moskov. Mat. Obšč. **5** (1956), 455–482. (Russian)

The article has three chapters. The first chapter discusses the region of attraction of the trivial solution of the system of equations

$$\text{(I)} \quad \frac{dx_i}{dt} = \sum_{k=1}^{n} a_{ik}x_k + f_i(t, x_1, \cdots, x_n) \quad (i = 1, 2, \cdots, n).$$

Let $\lambda_i$ be a root of the characteristic equation

$$\det \|A - \lambda E\| = 0$$

and let the following conditions be fulfilled: (1) $\mathrm{R}\lambda_i \leqq -b^2 \neq 0$; (2) in the sphere $\|x\|^2 = \sum_{i=1}^{n} x_i^2 \leqq R^2$ we have

$$|f_i(t, x_1, \cdots, x_n)| \leqq C_R\|x\|,$$

where the $C_R$ are constants depending only on $R$; (3) the functions $f_i(t, x_1, \cdots, x_n)$ are such that there exist solutions of system (I) passing through an arbitrary point of the sphere $\|x\| \leqq R$ and defined for all those values of $t$ for which the norm of the solution is less than $R$.

Condition (3) is fulfilled if the $f_i$ are continuous in their arguments, but the author does not assume continuity and uniqueness of the solution. Theorem 1: If for all $x$ for which $\|x\| \leqq R$, the constant $C_R < b^2n^{-2}m^{-1/2}\varepsilon_{\max}^{n-1}$, where $\varepsilon_{\max} = (n-1)^{1/2}b^2n^{-2}\|A\|$ and $m$ is the number of functions $f_i$ which are different from zero, then all integral curves of the system (I) issuing from points of the sphere $\|x\| \leqq \rho_R = \varepsilon_{\max}^{n-1}R$ will be $0^+$-curves (that is, $\|x\| \to 0$ as $t \to +\infty$) located for $t \geqq 0$ in the sphere $\|x\| \leqq R$.

Corollary: If $C_R < b^2n^{-2}m^{-1/2}\varepsilon_{\max}^{n-1}$ for all $x$, then the system (I) is asymptotically stable in the large. {Reviewer's remark: System (I) considered here is more general, in accordance with condition (3), than the system considered by P. V. Atrašenok [Vestnik Leningrad. Univ. **9** (1954), no. 8, 79–106; MR **17**, 1088] and I. G. Malkin [Prikl. Mat. Meh. **16** (1952), 495–499; MR **14**, 170], and the estimate of V. V. Nemyckiĭ is derived in a simpler way than by these authors, although the estimate of P. V. Atrašenok will in some cases be more accurate.}

Theorem 1 not only estimates the region of attraction in its dependence on $C_R$, but also indicates the region $R$ which will not be left by a moving point originating in the region $\rho_R$. The author further takes note of those special cases of system (I) where the estimates obtained can be improved. Theorem 1 is applied to find the region of stability of systems of automatic regulation with an arbitrary number of regulators and to find the region $R$ which will not be left by motions beginning in the region $\rho_R$ in the case that the system (I) is linear.

The second chapter considers system (I) under the assumption that the corresponding linear system has characteristic roots of opposite signs. For such systems a sphere is found (theorem 5), centered on the origin, in

which motions, with the exception of the null solution, do not remain for $-\infty < t < \infty$. On this basis the region of repulsion is indicated in the case that the real parts of all roots of the characteristic equation are positive.

In the third chapter the author studies the oscillatory regimes of a system of differential equations of the form

$$\text{(II)} \qquad \frac{dx_i}{dt} = f_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n).$$

The definition is introduced of a toroidal set $G$ in a $k$-dimensional space as the homomorph of a certain topological product of a circumference with a closed $(k-1)$-dimensional region, with the possible exclusion of a finite number of interior regions. The following definition of an oscillatory regime is reached: Let $f(p, t)$ be a recurrent trajectory completely contained in a toroidal region $G$ and let there exist a number $T_0$ such that in the interval of time $0 \leqq \tau \leqq \tau_p \leqq T_0$ the point $q = f(\bar{p}, t)$, where $\bar{p}$ is an arbitrary point $f(p, T_0)$, passes once and only once through each topological meridional section; then the trajectory $f(p, t)$ is called an oscillatory regime with respect to the toroidal region $G$. For the study of oscillatory regimes the author proposes the method of rotating functions of Lyapunov. The method runs essentially as follows. Through all boundary points of the toroidal region $G$ let all trajectories of the system (II) be contained within $G$. Let there exist a function $V(x_1, \cdots, x_n)$, defined on the torus, such that its derivative with respect to $t$, in virtue of system (II), is greater than a certain positive number. Further let the function $V$ be such that the motion "of all trajectories within $G$" is guaranteed to be "in the same direction". Then system (II) has, within the torus $G$, an oscillatory regime (theorem 6). It is shown how, with the aid of the function $V$, it is possible to estimate the "frequency of the oscillations."

Further, on the basis of the well-known Brouwer fixed-point theorem, the author establishes a criterion of sufficiency for existence of a periodic motion for system (I) (theorem 7 and corollary).

*N. P. Erugin* (RŽMat **1957** #2298)

803:

**Colombo, Giuseppe. Sulla determinazione analitica delle soluzioni periodiche dei sistemi non-lineari autonomi.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **26** (1959), 662–664.

The author considers the equation

$$\text{(*)} \qquad \alpha(x)\ddot{x} + \tfrac{1}{2}\alpha'(x)\dot{x}^2 + g(x) = \varepsilon f(x, \dot{x}, \varepsilon), \quad \alpha(x) \neq 0,$$

making the assumption that in the plane $(x, \dot{x})$ the trajectories of $\alpha(x)\ddot{x} + \frac{1}{2}\alpha'(x)\dot{x}^2 + g(x) = 0$ which meet the interval $x_1 \leqq x \leqq x_2$ are all closed curves, each one of which meets the $x$-axis at another point. In terms of maximum or minimum of a certain integral he then gives a criterion about the existence, for small $\varepsilon$, of a stable or unstable limit-cycle of (*).

*M. M. Peixoto* (Rio de Janeiro)

804:

**Filimonov, Yu. M. On the stability of solutions of differential equations of second order.** Prikl. Mat. Meh. **23** (1959), 596–598 (Russian); translated as J. Appl. Math. Mech. **23**, 846–849.

For the study of the stability of the origin for the differential system $dx/dt = X(x, y)$, $dy/dt = Y(x, y)$, the author supposes that $n$ functions $V_j(x, y)$, $(x, y) \in G_j$, $j = 1, \cdots, n$, are given, each $V_j$ being continuous and single valued in a region $G_j$ of the $xy$-plane. He supposes that any two consecutive regions $G_1, G_2, \cdots, G_n, G_1$ contain a well determined simple line passing through the origin. These curves are said to be the $L$-lines. The functions $V_j$ are supposed to be positive in a neighborhood of the origin (except at the origin where they vanish), and to vary strictly monotonically along the bounding curves. It is assumed that starting from an arbitrary point $A$ on one of the lines and within $U$, and having selected one sense of the rotations around the origin, it is possible to construct a continuous path made up of pieces along which the corresponding function $V_j$ is constant. This path will intersect the initial $L$-curve at a point $B$ in general distinct from $A$. If we assume that within $U$ this coincidence is impossible, then we always have $V(A) > V(B)$ or always $V(A) < V(B)$, where $V$ is any of the $V_j$ defined on the initial $L$-curve. The use of these $n$ functions $V_j$ as Lyapunov functions leads to stability and instability theorems which are analogous to the classical ones of Lyapunov.

*L. Cesari* (Ann Arbor, Mich.)

805:

**Rabinovič, Yu. L.; Hapaev, M. M. Linear equations involving a small parameter in the highest derivative in the neighbourhood of a regular singular point.** Dokl. Akad. Nauk SSSR **129** (1959), 268–271. (Russian)

Nel campo complesso è data l'equazione

$$(1) \qquad \bar{L}[w] = \varepsilon z^m w^{(m)} + \sum_{k=0}^{\infty} \varepsilon^k \bar{L}_k[w] = 0,$$

dove $\bar{L}_k[w] = \sum_{s=0}^{m-1} z^s \bar{q}_{k,s}(z) w^{(s)}$, essendo i coefficienti $\bar{q}_{k,s}(z)$ olomorfi per $z = 0$ ed essendo $\bar{q}_{0,m-1}(0) \neq 0$. Posto $Q_s(\varepsilon) = \sum_{k=0}^{\infty} \bar{q}_{k,s}(0)\varepsilon^k$ l'equazione caratteristica associata alla (1) è la

$$(2) \qquad \varepsilon\rho(\rho-1)\cdots(\rho-m+1) + \sum_{s=0}^{m-1} Q_s(\varepsilon)\rho(\rho-1)\cdots(\rho-s+1) = 0$$

mentre quella associata alla equazione $\bar{L}_0[w] = 0$ è la

$$(3) \qquad \sum_{s=0}^{m-1} Q_s(0)\rho(\rho-1)\cdots(\rho-s+1) = 0.$$

Sia $\rho_1$ una radice semplice della (3) e sia $\rho_1(\varepsilon)$ quella radice semplice della (2) tale che $\rho_1(\varepsilon) \to \rho_1$ per $\varepsilon \to 0$; allora la sostituzione $w = z^{\rho_1(\varepsilon)} u$ muta le soluzioni $w$ della (1) nelle soluzione $u$ della

$$(4) \qquad L[u] = \varepsilon z^m u^{(m)} + \sum_{k=0}^{\infty} \varepsilon^k L_k[u] = 0,$$

dove $L_k[u] = \sum_{s=1}^{m-1} z^{k-1} q_{k,s}(z) u^{(s)} + q_{k,0}(z) u$. Sia

$$(5) \qquad \bar{u} = \sum_{h=0}^{\infty} \varepsilon^h u_h(z)$$

soluzione formale della (4) cosicchè le $u_h$ sono definite dalle relazione ricorrenti $L_0[u_0] = 0$, $L_0[u_{h+1}] + \{L_1[u_h] + z^{m-1} u_h^{(m)}\} + L_2[u_{h-1}] + \cdots + L_{h+1}[u_0] = 0$ e dalle condizioni $u_0(0) = 1$, $u_{h+1}(0) = 0$ $(h = 0, 1, 2, \cdots)$. Gli Autori provano che la soluzione formale (5) è lo sviluppo asintotico della soluzione della (4) $u(z, \varepsilon)$ regolare in $z = 0$ e tale che $u(0, \varepsilon) = 1$ purchè $|z|$ risulti $<$ di una certa costante ed $\varepsilon \to 0$ restando nella regione che si ottiene escludendo dal

piano complesso $\varepsilon$ un certo angolo di vertice 0. Un'applicazione viene fatta all'equazione $\varepsilon z^2 w'' + z\varphi_1(z, \varepsilon)w' + \varphi_2(z, \varepsilon)w = 0$ con $\varphi_1$, $\varphi_2$ olomorfe per $z = 0$, $\varepsilon = 0$ e con $\varphi_1(0, 0) \neq 0$. *R. Conti* (Firenze)

806:

**Langer, Rudolph E. Formal solutions and a related equation for a class of fourth order differential equations of a hydrodynamic type.** Trans. Amer. Math. Soc. **92** (1959), 371–410.

The differential equations discussed are of the type

$$(1)\quad \frac{d^4w}{dz^4} + \lambda^2\{P(z, \lambda)\frac{d^2w}{dz^2} + Q(z, \lambda)\frac{dw}{dz} + R(z, \lambda)w\} = 0,$$

where $\lambda$ is a large complex parameter, and the concern is with asymptotic expressions for a fundamental system of solutions as $\lambda \to \infty$. It is assumed that $P$, $Q$, and $R$ are holomorphic in both variables in a neighborhood of $z = 0$, $\lambda = \infty$. The further assumption that $P_0(z) = P(z, \infty)$ has a first order zero at $z = 0$ makes this point a so-called turning point of the differential equation. The author [same Trans. **84** (1957), 144–191; MR **18**, 738] described a method for calculating certain asymptotic expansions of solutions valid in regions of the $z$-plane that contain this turning point. That method did not apply to a small but important subclass of equations of type (1). The alternative treatment given in the present paper does not suffer from such restrictions. In addition, it is simpler and more direct.

In the first part of the paper the author studies formal solutions of the form $\sum_{n=0}^{\infty} U_n(z)\lambda^{-n}$. He shows that there always exists at least one linearly independent such series with all $U_n(z)$ holomorphic at $z = 0$. The construction of further solutions is based on the auxiliary differential equation

$$(2)\quad \frac{d^3v}{dx^3} + \lambda^2 x\frac{dv}{dx} - \lambda^2\mu v = 0,$$

in which $\mu$ is a parameter, whose asymptotic theory was given by the author in Duke Math. J. **22** (1955), 525–542 [MR **17**, 365]. The solutions of (1) and (2) are related in the paper by setting $x = (\frac{3}{4}\int_0^z P_0^{1/2}dz)^{2/3}$,

$$w(z, \mu) = A(z, \nu)v(x, \mu) + \lambda^{-1}B(z, \nu)\frac{dv(x, \mu)}{dx} + \lambda^{-2}C(z, \nu)\frac{d^2v(x, \mu)}{dx^2}.$$

By means of a recursive algorithm, formal series

$$\nu = \sum_{n=0}^{\infty}\frac{\nu_n}{\lambda^n}, \qquad A(z, \nu) = \sum_{n=0}^{\infty}\frac{\alpha_n(z)}{\lambda^n},$$

$$B(z, \nu) = \sum_{n=0}^{\infty}\frac{\beta_n(z)}{\lambda^n}, \qquad C(z, \nu) = \sum_{n=0}^{\infty}\frac{\gamma_n(z)}{\lambda^n},$$

with $\alpha_n(z)$, $\beta_n(z)$ and $\gamma_n(z)$ holomorphic at $z = 0$, are determined such that the corresponding functions $w(z, \mu)$ solve the differential equation (1) in the formal sense, if $v$ is replaced by certain asymptotically known solutions of (2).

In most cases this method yields directly four formal solutions of (1) that are formally linearly independent. However, there is an exceptional class of differential equations of type (1), called "distinctive" by the author, for which the method so far described supplies only three linearly independent solutions. This case is dealt with by performing a suitable perturbation of the given equation and then letting the perturbation term tend to zero.

The proof that the formal series solutions obtained have asymptotic character is only partially carried out in the paper. *W. Wasow* (Madison, Wis.)

807:

**Kuzmak, G. E. Asymptotic solutions of nonlinear second order differential equations with variable coefficients.** Prikl. Mat. Meh. **23** (1959), 515–526 (Russian); translated as J. Appl. Math. Mech. **23**, 730–744.

A technique developed by N. M. Krylov and N. N. Bogolyubov [*Issledovanie prodol'noĭ ustoĭčivosti aeroplana*, Gos. Aviac. i Avtotpakt. Izd.-vo, Moscow-Leningrad, 1932] and the author [Ukrain. Mat. Ž. **10** (1958), no. 2, 128–146; Prikl. Mat. Meh. **21** (1957), 262–271; MR **20** #2516; **19**, 651] for certain quasilinear systems is applied to the computation of the principal terms of the formal asymptotic expansion of solutions of the real nonlinear equation

$$(1)\quad \frac{d^2y}{dt^2} + \varepsilon f(\tau, y)\frac{dy}{dt} + F(\tau, y) = 0,$$

where $\tau = \varepsilon t$ and $\varepsilon$ is a small positive parameter, in powers of $\varepsilon$. A solution of (1) is sought in the form $y = y(\tau, \omega)$ with (2) $d\tau/dt = \varepsilon$, $d\omega/dt = \phi(\tau)$, where $\phi$ is a non-negative function and where $0 \leq t \leq \tau_0/\varepsilon$. This requirement transforms (1) to the partial differential equation

$$(3)\quad \phi^2(\tau)\frac{\partial^2y}{\partial\omega^2} + F(\tau, y) + \varepsilon\left[2\phi(\tau)\frac{\partial^2y}{\partial\omega\partial\tau} + \phi'(\tau)\frac{\partial y}{\partial\omega} + f(\tau, y)\phi(\tau)\frac{\partial y}{\partial\omega}\right] + \varepsilon^2\left[\frac{\partial^2y}{\partial\tau^2} + f(\tau, y)\frac{\partial y}{\partial\tau}\right] = 0.$$

Requiring (4) $y(\tau, \omega) = y_0(\tau, \omega) + \varepsilon y_1(\tau, \omega)$ to satisfy (3) up to terms of order $\varepsilon^2$ leads to a pair of equations for $y_0$ and $y_1$. It is shown that if $y_0$ and $y_1$ can be determined as periodic functions of $\omega$, of period $T\omega$ independent of $\tau$, and if $f$, $F$, $\phi$ are sufficiently smooth, then (4) satisfies (1) up to terms of order $\varepsilon^2$. The requirement that $y_0$ be periodic in $\omega$ is equivalent to requiring that (1) has for $\varepsilon = 0$ a family of periodic solutions. The technique is applied to the equation $d^2y/dt^2 + a(\tau)dy/dt + b(\tau)y^3 = 0$, for which the equation determining $y_0$, called the "standard" equation, can be integrated explicitly by means of Jacobi elliptic functions. An approximation procedure using trigonometric polynomials, for those cases where the "standard" equation cannot be integrated explicitly, is also considered. *J. A. Nohel* (Atlanta, Ga.)

808:

**Halanay, Aristide. Solutions périodiques des systèmes linéaires à argument retardé.** C. R. Acad. Sci. Paris **249** (1959), 2708–2709.

Consider the equation

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) + f(t),$$

where $A$, $B$, and $f$ are periodic of period $\omega > \tau$. A proof is sketched that this equation has a periodic solution $x$ of period $\omega$ if and only if the homogeneous equation ($f = 0$) has no non-null solution. *E. Pinney* (Berkeley, Calif.)

## PARTIAL DIFFERENTIAL EQUATIONS

See also 880, 896, B1088, B1331.

809:

**Moisil, Gr. C. Sur les fonctions monogènes au sens de Feodorov.** Bŭlgar. Akad. Nauk. Izv. Mat. Inst. **4**, no. 1, 25–34 (1959). (Bulgarian and Russian summaries)

The author considers the system of $m$ differential equations with constant coefficients in $n$ independent variables and with $r$ unknown functions: $\sum_{k,s} a_{iks}\partial\varphi_k/\partial x^s = 0$ ($k=1, \cdots, r$; $s=1, \cdots, n$; $i=1, \cdots, m$). He examines cases where constants $C_{ik\xi}$ exist (not all zero) such that $(\varphi_1{}^1, \cdots, \varphi_r{}^1)$ and $(\varphi_1{}^2, \cdots, \varphi_r{}^2)$ are two sets of solutions then so also is

$$\varphi_i = \sum_{k\xi} C_{ik\xi}\varphi_k{}^1\varphi_\xi{}^2 \quad (i = 1, \cdots, r).$$

The system giving conditions for monogeneity in the sense of Fedorov [Dokl. Akad. Nauk SSSR (N.S.) **48** (1945), 389–390; MR **8**, 25] is one system of the indicated type.

*P. Civin* (Eugene, Ore.)

810:

**Hong, Imsik. On the equation $\Delta u+\lambda f(x, y)=0$ under the fixed boundary condition.** Kōdai Math. Sem. Rep. **11** (1959), 95–108.

Continuous dependence of eigenvalues and eigenfunctions upon the domain $D$ is demonstrated for the equation $\Delta u+\lambda f(x, y)u=0$ in the plane. It is assumed that $f(x, y)$ has continuous first derivatives and satisfies $M \geqq f(x, y) \geqq \mu > 0$ for some constants $M$ and $\mu$. It is shown that if $D$ is bounded and a sequence $D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots$ of domains with piecewise smooth boundaries $C_n$ exhausts $D$, then the first eigenvalues $\lambda_{1,n}$ of the problems $\Delta u+\lambda f u=0$ in $D_n$, $u=0$ on $C_n$, approach a limit $\lambda_1$, which is independent of the exhausting sequence. If the first eigenfunctions $u_{1,n}$ are normalized by $\iint_{D_n} f(x, y)u_{1,n}^2 d\sigma = 1$, they converge in $D$ to a limit function $u_1$ which is also independent of the exhausting sequence. Moreover, $\Delta u_1+\lambda_1 f u_1=0$ and $u=0$ on the boundary of $D$ save for the exceptional points of the Green's function for $D$. A similar result holds for the other eigenvalues and eigenfunctions with the exception that the limit "eigenfunction" is not, in general, independent of the choice of the exhausting sequence. Previous results of this nature were obtained by the author for the equation $\Delta u+\lambda u=0$ [same Rep. **9** (1957), 179–190; MR **20** #162]. *M. Schechter* (New York)

811:

**Maurin, K. Eine Abschätzung für Eigenfunktionen. Beschränktheit der Eigenfunktionen der Verschiebung invarianter Operatoren auf homogenen Räumen.** Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 337–341. (Russian summary, unbound insert)

Let $A$ be a self-adjoint operator on $L^2(\Omega, \sigma)$ such that for some essentially non-vanishing function $h$, $B=h(A)$ has a Carleman kernel $b(x, y)$, $c^2(x)=\int_\Omega |b(x, y)|^2 d\sigma(y) < \infty$. Using Gårding's theory of eigenfunction expansions, the author shows that, for almost every $\lambda$,

$$\sum_{k=1}^{N(\lambda)} |e_k(x, \lambda)|^2 \leq M(\lambda)c^2(x)$$

for some constant $M(\lambda)$, where the $e_k(x, \lambda)$ are the eigenfunctions of $A$. In case $c^2(x)$ is bounded this implies that the eigenfunctions are bounded, giving a theorem of A. G. Kostyučenko [Dokl. Akad. Nauk SSSR **114** (1957), 249–251; MR **19**, 872]. {There is a misprint in Lemma 2: for $L^2(\Omega, \sigma)$ read $L^1(\Omega, \sigma)$.}

The author applies this result to invariant self-adjoint operators $A$ of the above type on homogeneous spaces (Theorem 3), showing that the eigenfunctions are bounded. The invariant partial differential operators on a weakly symmetric Riemannian manifold have bounded eigenfunctions (Theorem 4). {Theorem 5 stands in need of correction, which is supplied in a subsequent paper of the author [same Bull. **8** (1960), 3–9; Satz 2].}

*E. Nelson* (Princeton, N.J.)

812:

**Bureau, Florent. Sur la représentation asymptotique de la fonction spectrale des opérateurs elliptiques du second ordre.** C. R. Acad. Sci. Paris **249** (1959), 1071–1073.

Soit

$$L_x = a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + b^i(x)\frac{\partial}{\partial x_i} + c(x)$$

un opérateur elliptique autoadjoint attaché à la métrique (symétrique définie positive) $ds^2 = a_{ij}(x)dx^i dx^j$ d'un espace de Riemann $R_p$, et soit $D$ un domaine borné de $R_p$. Soient $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k \to \infty$ les valeurs propres du problème $L_x u+\lambda u=0$ et $u(x)=0$ sur Fr $D$. Soient $u_k(x)$ ($k=1, 2, \cdots$) les fonctions propres correspondantes et soit $\theta(x, y, \lambda)$ la fonction spectrale de $L_x$, c'est-à-dire $\theta(x, y, \lambda) = \sum_{\lambda_n < \lambda} u_n(x)u_n(y)$ si $\lambda \geqq \lambda_1$, $=0$ si $0 \leqq \lambda < \lambda_1$. En utilisant une méthode ayant comme point de départ une idée de B. M. Levitan [Mat. Sb. (N.S.) **35** (**77**) (1954), 267–316; MR **17**, 158] et un théorème taubérien correspondant, on obtient pour $\lambda \to \infty$ la représentation asymptotique

$$\theta(x, x; \lambda) \sim \frac{A(x)}{\pi^{p/2}2^p\Gamma(\frac{1}{2}p+1)}\lambda^{2/p}$$

où $A(x)=|a_{ij}(x)|^{1/2}$. *C. Foiaş* (Bucharest)

813:

**Eicker, Friedhelm. Über eine Rand- und Mittelwertaufgabe der Poissonschen Differentialgleichung.** Z. Angew. Math. Mech. **39** (1959), 279–284. (English. French and Russian summaries)

Si consideri nel piano $(x, y)$ un dominio regolare $D$ e nello spazio $(x, y, z)$ il cilindro retto $T$ di base $D$ e altezza $z_0$: $(x, y) \in D$, $0 \leqq z \leqq z_0$. Viene studiato il seguente problema relativo all'equazione di Poisson: determinare $w(x, y, z)$ tale che (1) $\Delta w = f$ nell'interno di $T$;

$$(2) \qquad \iint_D w(x, y, z)dxdy = 0$$

per ogni $z$ di $(0, z_0)$; (3) $w$ assuma sulla superficie laterale $M$ di $T$ valori dipendenti solo da $z$ e nulli per $z=0$ e $z=z_0$ (cioè $w(x, y, z)=g(z)$ su $M$, con $g(0)=g(z_0)=0$ e $g(z)$ da determinarsi).

Nell'ipotesi che $f$ sia continua in $T$ e 2 volte derivabile rispetto a $z$ l'A. dimostra che esiste una e una sola soluzione del problema continua in $T$ e 2 volte derivabile nell'interno di $T$. Il problema è anzitutto ricondotto ad uno analogo in cui le condizioni (1) e (2) vengono rispettivamente sostituite con (1') $\Delta u=0 \ldots$, (2') $\iint_D u(x, y, z)\cdot dxdy = \gamma(z)$, con $\gamma(z)$ assegnata; e viene poi risolto mediante

uno sviluppo in serie del seguente tipo

$$u(x, y, z) = \sum_{\nu=1}^{\infty} u_\nu(x, y) \operatorname{sen} \frac{\nu\pi z}{z_0},$$

i cui coefficienti $u_\nu(x, y)$ si determinano risolvendo un opportuno problema relativo all'equazione metaarmonica.

Il problema viene infine impostato anche per insiemi più generali di $T$, ottenibili da $T$ medianti una opportuna deformazione, e ne viene dimostrata l'equivalenza con una equazione integrale lineare di I specie, ottenuta rappresentando le soluzioni di $\Delta w = f$ mediante la funzione di Green del relativo problema di Dirichlet.

*E. Magenes* (Pavia)

814:

**Burnat, M. Über partielle Differentialgleichungen vom elliptischen Typus mit singulären Koeffizienten.** Studia Math. **18** (1959), 137–159.

Let $A = \sum \partial_k a_{kj}(x)\partial_j$ $(j, k = 1, 2, 3;\ \partial_j = \partial/\partial x_j)$ be a real elliptic differential operator with smooth coefficients and let $a(x)$ be real and locally square integrable. Although unnecessarily burdened with considerations of selfadjointness, the paper deals with local regularity properties of solutions $u$ of $(A + a)u = 0$. Let $R$ be a spherical shell around a sphere $K$ and put $\|u\|^2 = \int_R |u(x)|^2 dx$. The main results of the paper are the statement that $u$ is continuous and the inequalities

$$|\operatorname{grad} u(x)| \leq C\|u\|\Big(1 + \int |x-y|^{-2}|a(y)|dy\Big),$$

$$|\operatorname{grad}^2 u(x)| \leq C\|u\|\Big(1 + \int |x-y|^{-2}(|a(y)| + |\operatorname{grad} a(y)|)dy\Big)$$

($(A + a)u = 0$ in $K \cup R$, $x \in K$, integration over $K$). These are immediate consequences of the formula

$$u(x) = \int B(x, y)u(y)dy + \int g(x, y)b(y)a(y)u(y)dy,$$

where $x \in K$, $z$ is a fundamental solution of $A$, $b \in C_0^\infty(K \cup R)$ equals 1 on $K$ and

$$B(x, y) = A_y\{g(x, y)\,(1 - b(y))\}$$

[Gårding, Kungl. Fysiogr. Sällsk. i Lund Förh. **20** (1950), 250–253; MR **12**, 708]. They are easily extended to the non-selfadjoint cases. Most of the author's questions about eigenfunction expansions are answered by N. Nilsson [ibid. **29** (1959), no. 1, 1–19; MR **21** #3652].

*L. Gårding* (Lund)

815:

**Kapilevič, M. B. Uniqueness theorems of singular Dirichlet-Neumann problems.** Dokl. Akad. Nauk SSSR **125** (1959), 23–26. (Russian)

Consider the equation (1) $u_{xx} + u_{yy} + a(r)u_y/y + F(r)u = 0$, $y > 0$; where $a(r) > 0$ and $F(r)$ $(r = (x^2 + y^2)^{1/2})$ are bounded continuous functions for $y > 0$, and, near $r = 0$,

$$a(r) = \sum_{s=0}^{\infty} a_s r^s, \quad F(r) = b_0 r^{-1} + \sum_{s=0}^{\infty} b_{s+1} r^s, \quad 0 < a_0 < 1.$$

There are two families of solutions $\{u\}$ and $\{\bar{u}\}$, characterized by (2) $u_\eta(x, 0) = 0$, $u(0, 0) \neq 0$ and (3) $\bar{u}(x, 0) = 0$, $\bar{u}_\eta(0, 0) \neq 0$, where $\eta = [y/(1 - a_0)]^{1-a_0}$. Let $M(r)$ be the solution of (4) $rM_{rr} + [1 + a(r)]M_r + rF(r)M = 0$, $M(0) = 1$, and let $\bar{M}(r)$ be the solution of (5) $r^2\bar{M}_{rr} + r\bar{a}(r)\bar{M}_r + \bar{F}(r)\bar{M} = 0$, $\bar{M}(0) = 1$, where $\bar{a}(r) = a(r) - 2a_0 + 3$, $\bar{F}(r) = r^2F(r) + (2 - a_0)[a(r) - a_0]$. The author states without proof:

$$M(r)u(0, 0) = D\int_0^\pi u(r\cos\theta, r\sin\theta)\sin^{a(r)}\theta\, d\theta$$

for any $u$ of $\{u\}$;

$$r^{1-a_0}\bar{M}(r)u_\eta(0, 0) = \bar{D}\int_0^\pi \bar{u}(r\cos\theta, r\sin\theta)\sin\theta\, d\theta.$$

Here $D$, $\bar{D}$ are constants depending only on $a_0$. He then states that the following uniqueness theorem follows as an application of the previous theorem: If $a(r) = a_0$, $F(r) \equiv -cr^{-1} - b^2 \leq 0$ in the domain $\Delta: y > 0$, $x^2 + \eta^2 < R^2$, then equation (1) plus either one of the boundary conditions (6) $u|_{r=R} = f$, $u_\eta(x, 0) = \nu$, (7) $\bar{u}|_{r=R} = \varphi$, $\bar{u}(x, 0) = \tau$, have a unique solution in $\Delta$ provided $c \neq 0$, $p \geq 1$ or provided $c = 0$, $p \geq 0$, where $p = 2a_0/(1 - a_0)$.

*A. Friedman* (Minneapolis, Minn.)

816:

**Žislin, G. M. A characteristic of the spectrum of Schrödinger's operator for systems of molecular type.** Dokl. Akad. Nauk SSSR **128** (1959), 231–234. (Russian)

The author continues an investigation of the spectrum of the non-relativistic Schrödinger operator for a system of $n$ particles which he has begun in his previous paper [same Dokl. **117** (1957), 931–934; MR **20** #1548]. Here the operator $H$ is a sum of two parts. The first part is linear in second derivatives of the form $\partial^2/\partial x_{ik}\partial x_{jk}$, $i, j = 1, 2, \cdots, n$, $k = 1, 2, 3$, where $\bar{x}_i = (x_{i1}, x_{i2}, x_{i3})$ is the position vector of the $i$th particle. The second part is $\sum V_{ij}(\bar{r}_{ij})$, where $\bar{r}_{ij} = \bar{x}_i - \bar{x}_j$. The author assumes that $V_{ij} \to 0$ as $\|\bar{r}_{ij}\| \to \infty$, that for two particles of like charge $V_{ij} \geq 0$, and $V_{ij} \leq 0$ otherwise. Furthermore, the weighted averages of $|V_{ij}|$ over any region $E \subset R_{3n}$ must satisfy certain linear inequalities with every weighting function $|\psi|^2$ such that $\psi$ has a generalized derivative (in Sobolev's sense) and $|\psi|$ and $|\operatorname{grad}\psi|$ are square integrable over $R_{3n}$. Assuming certain conditions on the (constant) coefficients in the first part of $H$, the author shows how by taking certain infima the number $\mu \leq 0$ can be obtained such that every $\lambda > \mu$ is a characteristic value of $\bar{H}$. (Operator $\bar{H}$ is a self-adjoint closure of $H$.) The condition for the existence of a discrete spectrum is given.

*Č. Masaitis* (Aberdeen, Md.)

817:

**Birman, M. Š. On the spectrum of Schrödinger and Dirac operators.** Dokl. Akad. Nauk SSSR **129** (1959), 239–241. (Russian)

A number of interesting results for the spectrum of the Schrödinger operator $A = -\Delta + p(x)$ in $m$-space are announced. Let $S$, $C$, and $N$ be the entire, continuous, and negative spectrum of $A$; and $S_\alpha$, etc., the same for $A_\alpha = -\Delta + \alpha p(x)$ where $\alpha > 0$. The function $p$ is said to be a $K$-potential if $\int_K |p(x)|dx \to 0$ as the cube $K$ tends to infinity and its size is fixed, and an $R$-potential if $p(x) = p(|x|)$ and $r\int_r^\infty |p(r)|dr \to 0$ as $r \to \infty$. Results: When $p$ is a bounded $K$-potential, $C$ is $\lambda \geq 0$. If $p(x) \leq 0$ for large $x$ and $N_\alpha$ is discrete for all $\alpha$, then $p$ is a $K$-potential. When $m = 1$ and $p$ is periodic, addition of a $K$-potential to $p$ does not change $C$. If $p(x) \leq 0$ for large $x$ and

$$\text{(1)} \qquad \sup R^{2-m}\int_{|x| \leq R} |p(x)|dx = \infty,$$

then $N$ is infinite. If $p(x)=p(|x|)\leqq 0$, $m\geqq 3$, and $N_\alpha$ is infinite for all $\alpha$, then (1) holds. Under the same assumption, $N_\alpha$ is finite for all $\alpha$ if and only if $p$ is an $R$-potential. If $m=1$, $p$ is periodic and $q\leqq 0$, then the spectrum of $A+\alpha q$ is discrete below $S$ for all $\alpha>0$ if and only if $q(x)$ and $q(-x)$ are $R$-potentials for $x>0$. When $m=1$, the number of elements of $N$ less than $-\varepsilon^2$ are majorized by $\int_0^\infty t[p(t)-\varepsilon^2]^+dt$ or $(2\varepsilon)^{-1}\int_0^\infty (1-\varepsilon^{-2\varepsilon t})p^+(t)dt$, where $\beta^+=(1+\operatorname{sgn}\beta)/2$. Finally, Dirac's operator $H$ with potential $q$ is studied. When $q=0$, its spectrum is $|\lambda|\geqq 1$. If $q$ is a $K$-potential, the continuous spectrum of $H$ is the same. If $A_\alpha$ with $p(x)=-|q(x)|$ has finite $N_\alpha$ for all $\alpha>0$, then $H$ has finite spectrum in $|\lambda|\leqq 1$. Estimates for the number of eigenvalues in $|\lambda|\leqq 1-\eta$ are given. No proofs.

*L. Gårding* (Lund)

818:

**Sears, D. B. Integrable square solutions of a partial differential equation.** Bul. Inst. Politehn. Iaşi (N.S.) **4** (8) (1958), no. 3–4, 11–18. (Russian and Romanian summaries)

In a previous paper [Canad. J. Math. **2** (1950), 314–325; MR **12**, 102], the author showed that, if $w=f(x, y, \lambda)$ is a solution of the equation

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}+\{\lambda-q(x, y)\}w = 0$$

which is of integrable square over the whole plane $D$ and satisfies certain conditions of regularity; and if $q(x, y)\geqq -Q(r)$, $r^2=x^2+y^2$, where $Q(r)$ satisfies certain conditions of which the essential one is

$$\int_0^\infty \{Q(r)\}^{-1/2}dr = \infty;$$

then

$$\iint_D (|f_x|^2+|f_y|^2)\{Q(\varepsilon r)\}^{-1}dxdy$$

is convergent ($\varepsilon=1$ or $\varepsilon>1$ in two cases). In the present paper these results are extended to solutions of the corresponding differential equation with a function $g(x, y)$ on the right-hand side, and to regions other than the whole plane. The results have applications to the problem of the uniqueness of the corresponding Green's function.

*E. C. Titchmarsh* (Oxford)

819:

**Aniščenko, R. I. Singular positive solutions of some elliptic equations.** Dokl. Akad. Nauk SSSR **129** (1959), 967–970. (Russian)

For suitable $f(V, M)$ the author proves by an iteration method the existence, for any prescribed $A>0$, of a unique positive solution to each of the problems, $\Delta V(M)=f(V(M), M)$, $\lim_{M\to 0}[V(M)r(M, 0)]=A$, $\partial V/\partial n|_S=0$ or $V|_S=0$, in a three dimensional region bounded by $S$. The problems are encountered in atomic physics. The assumptions on $f(V, M)$ are, essentially, that $\partial f/\partial V>0$, $\partial^2 f/\partial V^2\geqq 0$ for $V>0$, and $f(V, M)=O(V^{2-k})$, $\partial f/\partial V=O(V^{1-k})$ for fixed $M$, where $k>0$.

*R. Finn* (Stanford, Calif.)

820:

**Günther, Paul. Über einige spezielle Probleme aus der Theorie der linearen partiellen Differentialgleichungen zweiter Ordnung.** Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math. Nat. Kl. **102** (1957), no. 1, 50 pp.

The first result, of the several treated in this paper, is a contribution to a famous problem concerning the diffusion of waves. For Lorentz' equation $u_{tt}-\sum_{i=1}^n u_{x_ix_i}=0$ in an even number $n+1$ of independent variables, the value of a solution at any point $Q$ is determined, according to Huygens' principle, by the values of the solution and of its derivatives on the intersection of any space-like initial manifold with the characteristic cone that issues from $Q$. An analogous property, also known as Huygens' principle, obviously holds for any equation arising from that of Lorentz by changing coordinates, multiplying the equation by a factor, or multiplying $u$ by a factor. Noting this, Hadamard had raised the question as to whether all linear, hyperbolic, partial differential equations of second order, whose solutions obey Huygens' principle, are obtainable from Lorentz' equation in the manner described. For equations with constant leading coefficients, an affirmative answer has been given by M. Mathisson [Acta Math. **71** (1939), 249–282; see also Hadamard, Ann. of Math. (2) **43** (1942), 510–522; MR **4**, 45; and L. Ásgeirsson, Comm. Pure Appl. Math. **9** (1956), 307–326; MR **18**, 487] in the case $n=3$; and an answer in the negative by K. Stellmacher [Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. Math.-Phys.-Chem. Abt. **1953**, 133–138; MR **15**, 710], in the cases of odd $n>3$. Equations with variable coefficients in four dimensions ($n=3$) of the particular form

$$(*) \qquad u_{tt}-g^{ij}(x)\nabla_i\nabla_j u-A^i(x)\nabla_i u+C(x, t)u = 0$$

are considered in this paper, $\nabla_i$ denoting covariant differentiation with respect to $x^i$ in the Riemannian manifold $M$ defined by the positive definite metric $ds^2=g_{ij}dx^idx^j$. It is shown that if the solutions of such an equation are subject to Huygens' principle then the equation is indeed reducible to that of Lorentz, provided $\nabla_i\nabla^i R\leqq 0$, $R$ denoting scalar curvature in $M$. The argument leading to this conclusion is based on certain tensor relations following from Huygens' principle. These relations had been deduced by K. Stellmacher in an unpublished manuscript and subsequently were found by the author independently [same Ber. **100** (1952), no. 2; MR **14**, 282] from an analytical equivalent of Huygens' principle offered by M. Mathisson and fully proved by Ásgeirsson (unpublished) and the reviewer [Comm. Pure Appl. Math. **9** (1956), 391–402; MR **18**, 487]. Huygens' principle is not discussed directly for other hyperbolic equations than those of form (*), but a criterion is developed for reducibility to this form. Later sections of the paper deal with initial value problems of the form

$$(**) \qquad u_{tt}+f(t)u_t-\Delta_2 u = 0, \quad u(x, 0)=\phi(x), \quad u_t(x, 0) = 0,$$

$\Delta_2$ denoting Laplace's operator for an $n$-dimensional Riemannian manifold $M$ with positive metric. $M$ here is assumed completely harmonic in the sense that, from any point of $M$ as pole, there is a fundamental solution of $\Delta_2 v=0$ depending solely on the geodetic distance from the pole. (For $n=2$ or 3, this condition is equivalent to that of constant curvature of $M$.) It is then shown that, for suitably selected $f(t)=(n-1)/t+\cdots$, the solution of problem (**) is given by the (unweighted) mean of $\phi$ over geodetic spheres of radius $t$ and center $x$. Commutativity of $\Delta_2$ with the averaging operator is an important fact in the proof. For $n=2$ and 3, a sort of converse to the last proposition

is given asserting that, if (**) can be solved by a weighted mean of $\phi$ over geodetic spheres of radius $t$ and center $x$, then $M$ has constant curvature. The last section of the paper is devoted to some mean value theorems for the solutions of elliptic, self-adjoint, differential equations of second order. *A. Douglis* (College Park, Md.)

821:

**Bielecki, Adam; Kisyński, Jan. Sur le problème de E. Goursat relatif à l'équation $\partial^2 z/\partial x \partial y = f(x, y)$.** Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **10** (1956), 99–126 (1958). (Polish and Russian summaries)

The generalized problem of Goursat in the domain $D=\{(x, y)|g(x)<y<b, h(y)<x<a\}$ with data differentiable on $\Gamma_x=\{(x, y)|y=g(x)\}$ and on $\Gamma_y=\{(x, y)|x=h(y)\}$, and continuous on $\Gamma_x \cup \Gamma_y$, is shown equivalent to the determination of differentiable solutions of the functional equations (1) $G=(G\lambda)+\sigma$ on $[0, a]$ and (2) $H=(H\mu)+\tau$ on $[0, b]$. Here, $g \in C^1[0, a]$, $h \in C^1(0, b]$, $g(0)=h(0)=0$, $0 \leq g \leq b$, $0 \leq h \leq a$, $(hg)(x) \neq x$ on $(0, a]$; $\sigma \in C^1[0, a]$, $\tau \in C^1(0, b]$ are given functions associated with the initial data and the forcing term $f \in C^0(D)$; $\lambda=hg$, $\mu=gh$, and all products of functions stand for compositions. The problems of Picard ($g \equiv 0$) and of Darboux ($g \equiv 0$, $h \equiv 0$) are included as $\Gamma_x$ and $\Gamma_y$ need not be nowhere characteristic.

The authors show that (1) and (2) have at most one solution which is continuous at zero. Existence is established without further assumption in the special case $h \in C^1[0, b]$, $g'(0)h'(0)<1$, i.e., when $\Gamma_x$ and $\Gamma_y$ are not tangent at $(0, 0)$. In the first generalized case ($h \in C^1[0, b]$ but $g'(0)h'(0)=1$) existence follows under further restrictions: that the boundary conditions be continuable to a function defined on the rectangle $[0, a]\times[0, b]$ with Lipshitz continuous first derivatives, and that for all $n$, $|(\lambda^n)'| \leq M$ on $(0, a]$. In the second generalized case ($\lim_{y\to 0} h'(y)=\infty$) one also requires that for all $n$, $|(\mu^n)'| \leq M$, and $\lim_{y\to 0} yh(y)=0$. *I. I. Kolodner* (Albuquerque, N.M.)

822:

**Tersenov, S. A. On a hyperbolic equation degenerating at the boundary.** Dokl. Akad. Nauk SSSR **129** (1959), 276–279. (Russian)

The author studies Cauchy's problem for the equation $y^n u_{yy}-u_{xx}+au_y+bu_x+cu=f$ in the region $y>0$ under the assumption that $a$, $b$, $c$ and $f$ are real analytic. The case $a=b=c=0$ is well known. If $0<n<1$, the problem is regular. It is assumed that $1 \leq n<2$. In this case, the degree of singularity depends on $n$ and $a$. If, e.g., $n=1$ and $a(x, 0) \geq 1$ or $n>1$ and $a(x, 0)>0$, then there is a unique solution with given $u(0, x)$. In other cases, an expression involving $u_y$ must also be prescribed. *L. Gårding* (Lund)

823:

**Nirenberg, L. On elliptic partial differential equations.** Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 115–162.

This paper presents a series of expository lectures covering the construction of fundamental solutions for elliptic operators with constant coefficients, the inequalities that are necessary for Dirichlet's problem, a general Weyl lemma, estimates at the boundary and boundary problems in a half-space. *L. Gårding* (Lund)

824:

**Čžou, Yuĭ-lin'. Some problems for nonlinear elliptic and parabolic equations.** Sci. Record (N.S.) **3** (1959), 538–543. (Russian)

This paper summarizes some recent results on existence and uniqueness theorems for non-linear elliptic and parabolic equations, due to S. Bernstein, Leray and Schauder, Oleĭnik, Petrovskiĭ, Ladyženskaya and others. Theorems are carefully and succinctly stated, and brief indications of method of proof are given. *R. Finn* (Stanford, Calif.)

825:

**Campanato, Sergio. Sui problemi al contorno per sistemi di equazioni differenziali lineari del tipo dell'elasticità. I.** Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 223–258.

Soit $\Omega$ un ouvert de $R^n$; $u=(u_1, \cdots, u_n)$ désigne un élément de l'espace produit $\mathscr{D}'(\Omega)^n$, où $\mathscr{D}'(\Omega)$ désigne l'espace des distributions sur $\Omega$. Les problèmes aux limites variationnels sont construits avec des formes quadratiques en les $S_k u$, où les $S_k$ (opérateurs élémentaires) sont des systèmes différentiels de $\mathscr{D}'(\Omega)^n$ en lui-même. Il est important de se convaincre que les $S_k$ ne sont pas toujours des dérivations ordinaires, ce qui entraine notamment que les espaces de distributions "d'énergie finie" associés au problème variationnel ne sont pas des espaces de Sobolev. L'A. étudie dans ce cadre le système de l'élasticité (ou des généralisations naturelles de ce système); alors $S_k u=(\varepsilon_{1k}(u), \cdots, \varepsilon_{nk}(u))$, avec $\varepsilon_{ik}(u)=\frac{1}{2}(\partial u_i/\partial x_k+\partial u_k/\partial x_i)$. Soit $E(\Omega)^n$ l'espace des $u \in L^2(\Omega)^n$ avec $S_k u \in L^2(\Omega)^n$ (les dérivations étant prises au sens des distributions), muni de sa structure hilbertienne naturelle. L'A. étudie divers sous-espaces de $E(\Omega)^n$ (chap. I), sous-espaces qui traduisent les conditions aux limites stables. L'A. introduit à cet effet diverses conditions de régularité sur les ouverts $\Omega$. C'est le point essentiel de l'étude, les applications (chap. II) aux problèmes variationnels étant alors standard.

*J. L. Lions* (Nancy)

826:

**Reinhart, Bruce L.; Sternberg, Shlomo. Exterior differential systems and elliptic equations.** Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1522–1523.

It is stated without proof that systems of elliptic partial differential equations (in the sense of Douglis-Nirenberg) give rise to exterior differential systems which are globally in involution. It is also stated as a remark of M. Hausner and Sternberg (unpublished) that the second order parabolic, hyperbolic, and ultrahyperbolic equations give rise to exterior differential systems which are not in involution. The authors think that the above suggests some sort of equivalence between the notions of ellipticity and involution, and also the possibility of a non-analytic theory of exterior differential systems in involution.

*M. Kuranishi* (Nagoya)

827:

**Bouix, Maurice. Note sur une distribution résolvante de l'opérateur de Helmholtz.** J. Math. Pures Appl. (9) **38** (1959), 391–402.

Following H. Garnir, Bull. Soc. Sci. Liège, **20** (1951), 174–194; 271–296 [MR **13**, 243, 352], it is shown that the type of resolvent distribution there defined for differential operators, which is a distribution $T$ corresponding to an operator $D$ such that $DT=\delta$, does not suffice to give the

solution for systems of differential equations of mathematical physics, and that one is obliged for a system of operators $D_i$ to consider elementary solutions of diverse orders $p$; here a solution of order $p$ satisfies

$$D_i T_h = 0 \quad \text{if} \quad h \neq i,$$
$$= \partial^p \delta / \partial x^{\alpha_1} \cdots \partial x^{\alpha_n} \quad \text{if} \quad h = i,$$

where $\alpha_1+\alpha_2+\cdots+\alpha_n=p$. The physical example discussed is that of determining the system of electrical and magnetic sources which will give zero field inside a closed surface and outside the closed surface gives the same field as a given system of sources inside the surface.

*J. L. B. Cooper* (Cardiff)

828:

**Džavadov, M. G. Investigation of a mixed problem in a class of generalized functions.** Dokl. Akad. Nauk Azerbaĭdžan. SSR **14** (1958), 195–199. (Russian. Azerbaijani summary)

Par utilisation de résultats de régularité dans les problèmes mixtes hyperboliques, et d'une idée de Višik et Sobolev [Dokl. Akad. Nauk SSSR **111** (1956), 521–523; MR **20** #5351], l'A. étudie quelques problèmes mixtes bien posés à "intégrale de Dirichlet" non finie.

*J. L. Lions* (Nancy)

829:

**Babič, V. M. Elementary solutions of hyperbolic equations.** Dokl. Akad. Nauk SSSR **129** (1959), 479–481. (Russian)

The method for solving

$$P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) u = 0,$$

where $P$ is a homogeneous polynomial of degree $m$ with constant coefficients, is described by Gel'fand and Šilov [*Obobščennye funkcii i deĭstviya nad nimi*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958; MR **20** #4182; cf. p. 300]. The author shows that this method can be generalized to obtain a solution of $Lu=0$ with the boundary condition at $t=t_0$: $u=\partial u/\partial t=\cdots=\partial^{m-2}/\partial t^{m-2}=0$, $\partial^{m-1}u/\partial t^{m-1}=\delta(x-x^0)$. $L$ is a polynomial of degree $m$ in $\partial/\partial t$ and $\partial/\partial x_i$ whose coefficients are analytic functions. It is assumed that $L$ has $m$ distinct real characteristic functions $\gamma_j$ satisfying $\gamma_j|_{t=0}=(\omega, x-x^0)$ with $\|\omega\|=1$. Let $u_j=\sum_{k=0}^{\infty} u_{jk}(t,x) f_k(\gamma_j)$ where $u_j$ is a formal solution of $Lu=0$, $f_k(\xi)=\int f_{k-1}(\xi)d\xi$, and $f_0$ is such that $h=\sum_{j=1}^{m} u_j$ satisfies $Lh=0$ with the boundary conditions at $t=t_0$:

$$h = \frac{\partial h}{\partial t} = \cdots = \frac{\partial^{m-2} h}{\partial t^{m-2}},$$

$$\frac{\partial^{m-1} h}{\partial t^{m-1}} = C_m \delta^{(m-1)}((x-x^0, \omega)) \quad (m \text{ odd}),$$
$$= C_m' \frac{1}{(x-x^0, \omega)^m} \quad (m \text{ even}).$$

The numerical coefficient $C_m$ is defined by

$$\delta(x) = C_m \int_{\|\omega\|=1} \delta^{(m-1)}(x, \omega) d\Omega \quad (m \text{ odd}),$$

or

$$\delta(x) = C_m \int_{\|\omega\|=1} \frac{d\Omega}{(x, \omega)^m} \quad (m \text{ even}).$$

The solution of $Lu=0$ with the prescribed boundary conditions is given by $u=\int_{\|\omega\|=1} h d\Omega$.

*C. Masaitis* (Aberdeen, Md.)

830:

**Slobodeckiĭ, L. N. The fundamental solution and the Cauchy problem for a parabolic system.** Mat. Sb. (N.S.) **46** (**88**) (1958), 229–258. (Russian)

In this paper the existence of a fundamental solution matrix and the Ladyženskaya estimates for its derivatives obtained by S. D. Ĕĭdel'man [Mat. Sb. (N.S.) **38** (**80**) (1956), 51–92; MR **17**, 857] for systems of partial differential equations parabolic in the sense of Petrovskiĭ are shown to remain valid under weaker assumptions on the coefficients. After establishing certain additional properties of the fundamental solution, the Cauchy problem for systems $u_t - A(x, t, \partial/\partial x)u = f(t, x)$ is shown to be correctly set under the corresponding weaker conditions. Here $x$ is a point of $E_n$, $u$ a vector of functions, etc. Finally applications of these results are given to higher dimensional Markov processes.

The assumptions on the coefficients are such as to guarantee the adjoint operator to have coefficients which are uniformly Hölder continuous in $x$ in the strip $E_n \times [OT]$. In addition, the coefficients of the highest order terms are assumed uniformly continuous in $t$ in the strip. If $A(t, x, \partial/\partial x)$ is strongly elliptic the latter condition can be omitted. *A. N. Milgram* (Minneapolis, Minn.)

831:

**Ĕĭdel'man, S. D. Cauchy problem for parabolic systems with growing coefficients.** Dokl. Akad. Nauk SSSR **127** (1959), 760–763. (Russian)

Consider the system (in vector notation)

$$\text{(A)} \quad \frac{\partial u}{\partial t} = \sum_{|k| \leq 2b} p_k(t, x) D_x^k u + \sum_{|k| \leq 2b-1} p_k^*(t, x) D_x^k u,$$

where $x=(x_1, \cdots, x_n)$, $-\infty < x_i < \infty$, $0 \leq t \leq T$. Assume that (1) there exists a positive continuous function $f(x)$ such that $|D_x^s p_k(t, x)| \leq A f(x)^{2b-|k|}$, $|s| \leq 2b$; and $|p_k^*(t, x)| \leq A f(x)^{2b-|k|-\varepsilon}$, where $A$, $\varepsilon$ are positive constants; (2) $D_x^{2b} p_k$ are locally Hölder continuous in $x$; $Q_k(t, x) \equiv p_k(t, x) f(x)^{|k|-2b}$ are continuous in $t$, uniformly in $x$; and $p_k^*$ are locally Hölder continuous in $x$ and continuous in $t$; and (3) all the roots $\lambda = \lambda(\sigma, \mu, t, x)$ of $\det\{\sum_{|k| \leq 2b} Q_k(t, x) \times (i\sigma)^k \mu^{2b-|k|} - \lambda E\} = 0$, where $|\sigma|^2 + \mu^2 = 1$, $0 \leq t \leq T$, $|x| > R$ (for some $R > 0$), satisfy $\operatorname{Re} \lambda < -\delta < 0$ ($\delta$ constant). The author then asserts the existence of a fundamental matrix-solution $G$ of (A), satisfying

$$\text{(B)} \quad |D_x^m D_y^k G(t, \tau, x-\xi, y)| \leq C_{mk}(t-\tau)^{-(n+|m|)/2b} \times \exp\{-c_1 f(y)^{2b} - c|x-\xi|^{2b/(2b-1)}(t-\tau)^{-1/(2b-1)}\}.$$

{Reviewer's remark: The author probably intends to assert (B) only for $|m| + |k| \leq 2b$.} The method which the author uses is that of the parametrix, which he employed in constructing fundamental solutions for parabolic equations and systems with bounded smooth coefficients [Math. Sb. (N.S.) **38** (**80**) (1956), 51–92; same Dokl. **120** (1958), 980–983; MR **17**, 857; **21** #216]. As an application of (B), he derives results on the correctness of the Cauchy problem for some systems with unbounded smooth coefficients.

*A. Friedman* (Minneapolis, Minn.)

832:

**Pogorzelski, W. Sur la résolution du problème aux dérivées tangentielles pour l'équation parabolique par la méthode des approximations successives.** Ann. Polon. Math. **7** (1959), 93–112.

In a previous paper [Ann. Sci. École Norm. Sup. (3) **75** (1958), 19–35; MR **20** #4085] the author proved the existence of a solution of a boundary value problem with tangential derivatives for a quasi-linear equation of parabolic type by using Schauder's fixed point theorem. He now solves the problem again by the method of successive approximations.

The new method uses somewhat less general conditions, but has the advantage of actually giving the solution.

*E. T. Copson* (St. Andrews)

833:

**Čudov, L. A. Singularities in the solutions of linear partial differential equations with constant coefficients.** Dokl. Akad. Nauk SSSR **125** (1959), 504–507. (Russian)

Let $u$ be a solution of $Lu=0$, where $L$ is a linear partial differential operator of order $m$ in $n$ variables with constant coefficients. Let $O$ be a singular point of $u$ and let $u$ satisfy the following conditions: (1) in some open neighborhood $G$ of $O$, $u$ has continuous derivatives of order $p$, $p \geqq m$; (2) $u$ satisfies $Lu=0$ in the usual sense in $G$; (3) all derivatives of $u$ of order $m-1$ satisfy the condition $|D^{m-1}u| < A|x|^{-q}$, where $q$ is a non-negative integer, $A$ is a constant and $|x|=\sqrt{(x_1^2+x_2^2+\cdots+x_n^2)}$; (4) some derivative of $u$ of order $\mu$, $0 \leqq \mu \leqq p$, has a discontinuity at the point $O$. Under these conditions, the author shows that if $L$ is not elliptic and $p$ is sufficiently large, then $p-\mu<n-1$ for $n$ an odd number and $p-\mu<n$ for $n$ even. The bound on $p$ is

$$\begin{aligned} p &\geqq n-1 \quad \text{for } n \text{ odd if } m \leqq \tfrac{1}{2}n, \\ &\geqq n \quad \text{for } n \text{ odd if } m > \tfrac{1}{2}n, \\ &\geqq 0 \quad \text{for } n \text{ even.} \end{aligned}$$

*C. G. Maple* (Ames, Iowa)

834:

**Fage, M. K. Solution of a Cauchy problem by increasing the number of independent variables.** Mat. Sb. N.S. **46** (88) (1958), 261–290. (Russian)

L'A., proseguendo l'ordine di idee di una serie delle sue ricerche recenti [Dokl. Akad. Nauk SSSR **115** (1957), 874–877; ibid. **108** (1956), 780–783, 1022–1025; Mat. Sb. (N.S.) **45** (87) (1958), 281–322; MR **20** #1011b; **18**, 214, 580; **21** #1456], tra cui l'ultima nota menzionata ne costituisce il sunto dei risultati principali di cui lo sviluppo ne forma l'oggetto della presente, considera il problema dei valori iniziali per l'equazione (1) $AF(w,x)=H(w,x)$, con i dati di Cauchy nel piano $x=c$, ove se ne prescrivono funzioni regolari $F$, $F_x^1$, $\cdots$, $F_{x^{n-1}}^{(n-1)}$, e l'operatore

$$A \equiv -\frac{\partial^n}{\partial w^n}+\frac{\partial^n}{\partial x^n}+\sum_{k+m\leqq n-1} P_{km}(w,x)\frac{\partial^{k+m}}{\partial w^k \partial x^m},$$

essendovi variabile complessa $w=u+iv$, mentre $x$ ne è variabile reale. Si indica che il problema considerato è collegato con quello della trasformazione di un operatore lineare differenziale ordinario d'ordine $n$ in un altro operatore. I coefficienti ed il secondo termine dell'equazione (1) si suppongono continui rispetto a tutte le variabili nel dominio cilindrico $C=G\times(a,b)$, $a<c<b$ e regolari nel dominio $G$, per ogni $x$ fissato, ove si pone $w \in G$ e $x \in (a,b)$. Con ogni punto $P_0 \in C$ si collega la piramide caratteristica con vertice $P_0(w_0,x_0)$ ed avente per spigoli le rette caratteristiche dell'equazione (1): $w=w\pm t_i\varepsilon_i$, $x=x_0 \mp t_i$ ($i=1,2,\cdots,n$), e la base nel piano $x=c$ (i segni superiori corrispondono al caso $x_0 \geqq c$), ove $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ sono le radici dell'unità e $t_i$ è un parametro reale. La piramide considerata ne costituisce l'immagine per la trasformazione $w=w_0 \pm \sum_1^n t_i\varepsilon_i$, $x=x_0 \mp \sum_1^n t_i$ della piramide $n$-dimensionale

$$t_1 \geqq 0, t_2 \geqq 0, \cdots, t_n \geqq 0, \quad t_1+t_2+\cdots+t_n \leqq |x_0-c|,$$

che ne risulta caratteristica per l'equazione

$$\frac{\partial^n f}{\partial t_1 \partial t_2 \cdots \partial t_n} + \sum_{k=0}^{n-1} \sum_{i_1<i_2<\cdots<i_k} P_{i_1 i_2 \cdots i_k}(t) \frac{\partial^k f}{\partial t_{i_1} \partial t_{i_2} \cdots \partial t_{i_k}} = h(t), \tag{2}$$

ove

$$f = F(w_0 \pm \sum_1^n t_i\varepsilon_i;\ x_0 \mp \sum_1^n t_i).$$

Siccome per l'equazione (2) è applicabile il metodo delle approssimazioni successive, lo stabilirsi della corrispondenza tra le soluzioni del problema di Cauchy per l'equazione (2) e quelle dell'equazione (1), conduce l'A. alla costruzione delle soluzioni di quest'ultima. Nell'ipotesi di regolarità abbastanza elevata dei coefficienti, la soluzione ne può essere esplicitata, in dipendenza dei dati iniziali, tramite integrali estesi alla base della piramide caratteristica (cfr. la formula (56) relativa al teorema 9, p. 288).

*D. Mangeron* (Iași)

835:

**Fage, M. K. Integral representations of operator-analytic functions of one independent variable.** Trudy Moskov. Mat. Obšč. **8** (1959), 3–48. (Russian)

This paper is a continuation of same Trudy **7** (1958), 227–268 [MR **21** #2091]; we use here the notation of that review. Assume the operator $L$ defined in an interval of the real independent variable $x$ and $M$ in a domain of the complex independent variable $w$; the coefficients of $M$ are assumed to be analytic, those of $L$ continuously differentiable up to an order equal to its index. The purpose of the present paper is to find an integral representation for the isomorphism $T$ between the algebras $A_{L,x_0}$ and $A_{M,w_0}$; according to J. Delsarte and A. Ya. Povzner [see MR **21** #2091] this is closely related to the solution of Cauchy's problem for $MF(w,x)=LF(w,x)$. The method used was developed by the author [see preceding review]. The case $M=d^n/dw^n$ is especially considered and the formulas are explicitly written down for $n=2, 3, 4$. Other formulas using contour integration are also established. The results are so numerous and the formulas so complicated that it is impossible to summarize them in this review.

*J. L. Massera* (Montevideo)

836:

**Sobolev, S. L. The fundamental solution of Cauchy's problem for the equation**

$$\frac{\partial^3 u}{\partial x \partial y \partial z}-\frac{1}{4}\frac{\partial u}{\partial t} = F(x,y,z,t).$$

Dokl. Akad. Nauk SSSR **129** (1959), 1246–1249. (Russian)

The Cauchy problem for the title equation (1) was

shown to be well posed by Petrovskiĭ [Byull. Moskov. Gos. Univ. Ser. A **1** (1938), no. 7, 1–72; see § 6, p. 27]. The equation is of interest because it is not included in the types which have been extensively studied. The author constructs explicitly the fundamental solution for this problem. The structure of the left member of (1) implies that the fundamental solution has the form

$$G(x, y, z, t) = t^{-1}\Lambda(\xi) \equiv t^{-1}\Lambda(xyzt^{-1}),$$

where $\Lambda(\xi)$ satisfies $(\xi d/d\xi+1)(\xi d^2/d\xi^2+d/d\xi+\frac{1}{4})\Lambda=0$. The final result is $G(x, y, z, t)=0$ for $t<0$; $G(x, y, z, t)=-(2\pi t)^{-1}Y_0[(xyzt^{-1})^{1/2}]$ if $t>0$, $xyz>0$; $G(x, y, z, t)=(\pi^2 t)^{-1}K_0[(-xyzt^{-1})^{1/2}]$ if $t>0$, $xyz<0$. $Y_0$ and $K_0$ are standard Bessel functions. The proof is by direct verification. *W. S. Loud* (Madison, Wis.)

837:

**Ždanov, G. A. On the convergence of a modification of Galerkin's method.** Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 223–225. (Russian)

838:

**Saltykow, N. Recherches sur l'intégrabilité d'équations aux dérivées partielles du second ordre de Monge-Ampère.** J. Math. Pures Appl. (9) **38** (1959), 365–389.

Le but de la note est de compléter les recherches de G. S. S. Ludford [Pacific. J. Math. **5** (1955), 441–450; MR **17**, 271] et M. H. Martin [ibid. **3** (1953), 165–187; MR **14**, 1091] sur l'intégrabilité des équations de Monge-Ampère de la forme

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}-\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2+X(x)^2Y(y)^2=0, \tag{1}$$

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}-\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2+\lambda^2(x, y)=0. \tag{2}$$

L'auteur ramène le problème de la recherche des cas d'intégrabilité des équations (1) et (2) à l'intégration de deux systèmes d'équations linéaires aux dérivées partielles du premier ordre d'une fonction inconnue. En profitant de la méthode de A. N. Korkin d'intégration d'équations aux dérivées partielles du premier ordre, il donne tous les cas possibles des solutions des problèmes posés par M. H. Martin et G. S. S. Ludford. *B. S. Popov* (Skopje)

839:

**Gouyon, René. Sur un problème d'équations aux demi-dérivées partielles, introduit par les houles à tourbillon discontinu.** C. R. Acad. Sci. Paris **249** (1959), 1172–1174.

840:

**Višik, M. I.; Lyusternik, L. A. Certain questions concerning perturbations in boundary value problems for partial differential equations.** Dokl. Akad. Nauk SSSR **129** (1959), 1203–1206. (Russian)

The authors consider the following singular perturbation problem:

$$L_\varepsilon u \equiv (L_0+\varepsilon L_1+\cdots+\varepsilon^l L_l)u(x)=h(x), \tag{1}$$
$$\gamma_i u \equiv \partial^i u/\partial n^i|_\Gamma = 0 \quad (i=0, 1, \cdots, l),$$

where $x$ is a point in a domain $D$ of $r$-space, $\Gamma$ is the boundary of $D$, $L_0$ is an elliptic operator of second degree, $L_l$ is an operator of order at most $l+2$, and the operator of highest order, $L_{2l}$, is elliptic of order $2l+2$. Previous work on the problem is referred to in the authors' paper [Uspehi Mat. Nauk **12** (1957), no. 5 (77), 3–122; MR **20**, #2539].

The method is illustrated with the finite-dimensional algebraic analogue

$$A_\varepsilon x \equiv (A_0+\varepsilon A_1)x = b \quad (\det A_1 \neq 0). \tag{2}$$

If $A_0$ is nonsingular, $x$ is analytic in $\varepsilon$. If $A_0$ is singular, $x$ has a pole of order $n$ in $\varepsilon$. The order $n$ is finite and is the number of terms in the longest "Jordan chain" of auxiliary vectors $y_0, y_1, \cdots, y_{n-1}$ where $A_0y_0=0$, $A_0y_l=-A_1y_{l-1}$, and $A_0y=-A_1y_{n-1}$ has no solution. One can also characterize $n$ as the order of the largest submatrix in the Jordan canonical form of $A_0$ among the submatrices corresponding to $\lambda=0$. The coefficients of negative powers of $\varepsilon$ are linear combinations of the various vectors in the several Jordan chains.

For the differential operator (1), when $\lambda=0$ is in the spectrum for $\varepsilon=0$, the situation is shown to be the same. It is necessary to require that $\|L_\varepsilon^{-1}\| \leq A\varepsilon^{-n_1}$ for some $n_1$. If this is the case, then $u_\varepsilon$ will involve only a finite number of negative powers of $\varepsilon$, at most to $\varepsilon^{-n_1}$. There is a finite number of "auxiliary vectors" needed, which are involved in the coefficients of negative powers of $\varepsilon$. It is also necessary to include certain boundary-layer terms in the expansion.

The cases that $\lambda=0$ is not in the spectrum for $\varepsilon=0$, and that all the $L_l$ have order at most two, are considered briefly. No boundary layer terms are necessary in this latter case. It is stated that extensions to higher order $L_0$ involve no further difficulties. Proofs are only sketched or indicated. *W. S. Loud* (Madison, Wis.)

## POTENTIAL THEORY

See also 950.

841:

**Günter, N. M. ★Die Potentialtheorie und ihre Anwendung auf Grundaufgaben der mathematischen Physik.** B. G. Teubner Verlagsgesellschaft, Leipzig, 1957. x+342 pp. DM 18.00.

In der deutschsprachigen mathematischen Literatur fehlte seit Jahren ein Lehrbuch der klassischen Potentialtheorie, eine Lücke, die durch die Übersetzung des bekannten Werkes von N. M. Günter aus dem Russischen geschlossen wurde. Der russische Originaltext [*Teoriya potenciala i ee premenenie k osnovnym zadačam matematičeskoi fizike*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953; MR **16**, 357] ist in seinen Hauptteilen eine Übertragung des 1934 in der Sammlung Borel (in französischer Sprache) erschienenen Buches des Verfassers: *La théorie du potentiel et ses applications aux problèmes fondamentaux de le physique mathématique* [Gauthier-Villais, Paris, 1934]; er wurde von Ch. L. Smolitzki gegenüber der französischen Ausgabe mit einigen Änderungen und Ergänzungen versehen, um ihn dem gegenwärtigen Stand der Wissenschaft anzupassen. Die Leitung lag in den Händen von W. I. Smirnow.

Das Buch enthält im wesentlichen die klassische Theorie der Randwertaufgaben und die Eigenschaften

ihrer Lösungen, insbesondere bei Annäherung an den Rand. Die lückenlose, strenge Durchführung der Betrachtungen bringt es mit sich, daß diese für mehr an den Anwendungen interessierte Leser etwas schwer lesbar sein dürften. Um diesem Leserkreis die Arbeit zu erleichtern und die Zusammenhänge besser hervortreten zu lassen, wurden gewisse, besonders diffizile Beweise, in einen Anhang verwiesen.

Das Werk gliedert sich in folgende fünf grössere Abschnitte: I. Hilfssätze; II. Die Theorie des Potentials; III. Das Neumannsche und Robinsche Problem; IV. Das Dirichletsche Problem; V. Greensche Funktionen und ihre Anwendungen. In I. finden sich fundamentale Integralformeln und Sätze über Randflächen bzw. -kurven. II. beschäftigt sich mit den Raum- und Flächenpotentialen und ihren Ableitungen. Schwerpunkt von III. und IV. ist die Diskussion der den genannten Problemen entsprechenden Integralgleichungen. V. gibt u.a. Ausblicke auf das Eigenwertproblem der mehrdimensionalen Schwingungsgleichung. Den Abschluß bildet eine kurze, sehr lesenwerte Biographie N. M. Günters aus der Feder von W. I. Smirnow und S. L. Sobolew.

Referent bedauert, daß auf die potentialtheoretischen Arbeiten von L. Lichtenstein nicht eingegangen wird.

*K. Maruhn* (Giessen)

842:

**Šaškin, Yu. A. On the inverse problem of potential theory.** Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 45–46. (Russian)

It is a question of two bodies having the same exterior potential, one being of greater density. The author sketches proofs of two theorems, each to the effect that the region $D_1$ occupied by the denser body must be contained by the region $D_2$ occupied by the other. In the first result, the densities are uniform in each case, and $D_1$, $D_2$ are starlike with respect to the origin; the proof is given for the plane, but appears to apply more generally. The second theorem allows the densities to be variable, and relies on conformal mapping; the boundary of $D_1$ is to be an analytic curve. Precise conditions on the boundaries were not specified in the first result.

*F. V. Atkinson* (Toronto, Ont.)

843:

**Habetha, Klaus. ★Über Mittelwerte von Lösungen linearer partieller Differentialgleichungen 2. Ordnung.** Inauguraldissertation, Freie Universität Berlin, 1959. 84 pp.

Verf. betrachtet in erster Linie Lösungen von Differentialungleichungen der Form $\Delta u + a_1(x, y)u_x + a_2(x, y)u_y \geqq c(x, y)u$. Dabei werden $a_1$ und $a_2$ als zweimal stetig differenzierbar, $c$ als stetig vorausgesetzt. Es werden Konvexitätseigenschaften der Mittelwerte

$$\mu_k(r) = \int_0^{2\pi} [u^+(r\cos\varphi, r\sin\varphi)]^k f(r, \varphi)d\varphi$$

und $m_k(r) = [\mu_k(r)]^{1/k}$ $(k \geqq 2)$ bewiesen, wobei $f$ eine gewisse (von $a_1$ und $a_2$ abhängige) Gewichtsfunktion darstellt. Aus diesen Mittelwerteigenschaften schliesst Verf. auf das Hopfsche Maximumprinzip sowie (nach Einführung von Annahmen über das Verhalten von $a_1$ und $a_2$ im Unendlichen) auf Sätze vom Phragmén-Lindelöfschen Typ. Es gelingt dem Verf., diese Untersuchungen auf einen Ungleichungstyp mit einem singulären Koeffizienten (die Gleichung der axialsymmetrischen Potentiale als Spezialfall umfassend) auszudehnen. —Weitere Resultate derselben Art handeln von den Lösungen $u(x_1, \cdots, x_n)$ der Ungleichung

$$\sum_{i,j=1}^{n} (a_{ij}u_{x_i})_{x_j} + \sum_{i=1}^{n} \frac{b(r)}{r} x_i u_{x_i} \geqq \frac{C}{r^2} u,$$

wobei $r = (x_1^2 + \cdots + x_n^2)^{1/2}$. Dabei wird u.a. die Matrix $(a_{ij}(x_1, \cdots, x_n))$ folgendermassen eingeschränkt: Sie soll symmetrisch sein, die zugehörige quadratische Form positiv semidefinit, und es soll stets $\sum_{j=1}^{n} a_{ij}x_j = x_i$ gelten $(i = 1, \cdots, n)$. —Schliesslich betrachtet Verf. auch von zwei Veränderlichen abhängige Mittelwerte von Realteilen analytischer Funktionen zweier komplexer Variablen.— Das wesentliche Werkzeug ist die Methode von T. Carleman [C. R. Acad. Sci. Paris **196** (1933), 995–997] und A. Dinghas [Ann. Acad. Sci. Fenn. Ser. A. I no. 250/8 (1958); MR **20** #2547, und dort erwähnte Arbeiten].

*A. Huber* (Zürich)

844:

**Doob, J. L. Conditional Brownian motion and the boundary limits of harmonic functions.** Bull. Soc. Math. France **85** (1957), 431–458.

Soit $R$ un espace de Green de frontière positive. Dans un travail antérieur [Teor. Veroyatnost. i Primenen. **2** (1957), 3–33; MR **21** #5240] l'auteur a défini le mouvement brownien sur $R$. Il étudie ici des processus plus généraux, appelés $h$-processus ou mouvements browniens conditionnels: ce sont les processus markoviens stationnaires (à temps de survie $\leqq \infty$) ayant pour densité de transition la fonction $p^h$ définie par $p^h(t, \xi, \eta) = p(t, \xi, \eta)\, h(\eta)/h(\xi)$, où $p$ est la densité de transition du mouvement brownien sur $R$, et $h$ une fonction surharmonique $> 0$ donnée sur $R$. Il étend au cas des $h$-trajectoires diverses propriétés établies dans l'article cité (cas $h = 1$); en outre, si $h$ est la fonction de Green de pôle $\xi \in R$, presque toutes les $h$-trajectoires convergent vers $\xi$ (ceci, et le résultat analogue lorsque $h$ est harmonique positive minimale, expliquent le titre du travail). Ces trajectoires permettent de donner une définition probabiliste de la notion de limite fine (en un point de $R$ ou en un point-frontière minimal) de Brelot-Naïm [voir L. Naïm, Ann. Inst. Fourier, Grenoble **7** (1957), 183–281; MR **20** #6608]. Application est faite à l'étude locale et à la frontière des fonctions surharmoniques; en particulier on montre que si $u$ est surharmonique $\geqq 0$ sur $R$, et si $h$ est harmonique $> 0$, $u/h$ admet à la frontière une limite $h$-fine qui est finie presque partout (pour la mesure $h$-harmonique); ceci constitue une généralisation importante d'un théorème établi par L. Naïm [loc. cit.] par des méthodes non probabilistes.

Les outils essentiels sont les martingales, la propriété forte de Markov, et une représentation intégrale explicite de la solution du $h$-problème de Dirichlet [voir M. Brelot, J. Math. Pures Appl. (9) **35** (1956), 297–335; MR **20** #6607] donnée antérieurement par l'auteur [Illinois J. Math. **2** (1958), 19–36; MR **21** #5242].

*J. Deny* (Paris)

845:

**Doob, J. L. Boundary limit theorems for a half-space.** J. Math. Pures Appl. (9) **37** (1958), 385–392.

La théorie précédente est illustrée par l'exemple d'un demi-espace euclidien $R$; on peut alors expliciter la

densité de transition du mouvement brownien sur $R$ et la fonction de Green de $R$, et retrouver, comme cas très particulier, un théorème de J. Lelong-Ferrand [C. R. Acad. Sci. Paris **226** (1948), 1161–1163; MR **10**, 39].

*J. Deny* (Paris)

846:

**Anger, Gottfried. Ein funktionalanalytischer Aufbau der Potentialtheorie.** Wiss. Z. Tech. Hochsch. Dresden **8** (1958/59), 679–685.

Exposé d'une partie des idées de la thèse de l'auteur [Dresden, 1957; voir aussi *Séminaire de théorie du potentiel*, 1958, Faculté des Sciences de Paris, 1959]. Il est caractérisé par la considération systématique des potentiels continus, qui remplace avantageusement, dans certains questions, celle des potentiels d'énergie finie. Après un rappel sur la théorie newtonienne, on donne une axiomatique de la théorie générale du potentiel par rapport à un noyau symétrique sur un espace localement compact; on établit un théorème de convergence et on s'intéresse particulièrement au principe du balayage "faible", qui se distingue du principe du balayage ordinaire par le fait qu'on ne suppose rien sur le comportement du potentiel balayé en dehors de l'ensemble sur lequel on effectue le balayage; malheureusement la démonstration d'une partie du résultat essentiel concernant ce principe (Satz 12) est erronnée, la majoration (31) étant illusoire.

*J. Deny* (Paris)

847:

**Gendler, M. G. Mean value theorem for biharmonic functions.** Dokl. Akad. Nauk. SSSR **129** (1959), 1207–1210. (Russian)

The author gives a short formal proof that if $f(x, y)$ satisfies the bi-Laplace equation $\Delta^2 f(x, y) = 0$ then

$$4\pi f(x, y) = \int_0^{2\pi} [f(x + r\cos\theta, y + r\sin\theta) + f(x + ir\cos\theta, y + ir\sin\theta)]d\theta.$$

The main tools are representations of some one parameter groups of linear differential operators due to N. P. Romanoff [Ann. of Math. (2) **48** (1947), 216–233; MR **8**, 520].

*R. R. Goldberg* (Evanston, Ill.)

## FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 748.

848:

**Naftalevič, A. G. A system of two linear difference equations with constant coefficients.** Mat. Sb. (N.S.) **46** (88) (1958), 421–432. (Russian)

A necessary and sufficient condition is given for the existence of meromorphic solutions $f(z)$ of the simultaneous difference equations

$$\sum_{j=1}^{n} a_j f(z + j\alpha) = g(z), \quad \sum_{k=1}^{m} b_k f(z + k\beta) = h(z), \quad \Im\left(\frac{\alpha}{\beta}\right) \neq 0,$$

where $a_j$, $b_k$, $\alpha$ and $\beta$ are given complex constants and $g(z)$ and $h(z)$ are known meromorphic functions. Full details relating to the structure of such solutions in terms of elliptic functions are also presented.

The possibility of entire solutions, when $g(z)$ and $h(z)$ are entire functions, is examined. The rate of growth of such solutions is studied and asymptotic periods discussed.

*H. L. Turrittin* (Minneapolis, Minn.)

849:

**Charzyński, Zygmunt. Uniformisation des fonctions. Remarque sur les courbes planes.** Soc. Sci. Lettres Łódź. Bull. Cl. III **9** (1958), no. 8, 12 pp.

Let $f$ and $g$ be real valued functions, defined and possessing continuous derivatives on the respective closed real intervals $[a, b]$ and $[c, d]$, such that: (1) the derivatives $f'$ and $g'$ vanish at only finitely many points in these respective intervals; (2) at any points $\xi$ and $\eta$ where $f(\xi) = g(\eta)$, the derivatives $f'(\xi)$ and $g'(\eta)$ are not both zero; (3) the functions $f$ and $g$ attain their absolute minimal values at just the respective points $a$ and $c$, and their absolute maximal values at just the respective points $b$ and $d$ (though both functions may possess additional local maxima and minima); (4) $f(a) = g(c)$ and $f(b) = g(d)$. Under these conditions it is shown that there exist two "uniformizing" functions $u$ and $v$, defined and possessing continuous derivatives on a common closed interval $[O, T]$, which map this interval onto the respective intervals $[a, b]$ and $[c, d]$ in such a fashion that $u(0) = a$, $v(0) = c$, $u(T) = b$, $v(T) = d$, and $f(u(t)) = g(v(t))$ holds for all $t$ in the interval $[O, T]$. An application of this result to plane curves is given.

*T. A. Botts* (Charlottesville, Va.)

850:

**Kordylewski, J.; Kuczma, M. On the functional equation $F(x, \varphi(x), \varphi[f(x)]) = 0$.** Ann. Polon. Math. **7** (1959), 21–32.

The authors study the functional equation mentioned in the title, continuing a previous work of M. Kuczma [same Ann. **6** (1959), 281–287; MR **21** #5830] and extending and correcting results of T. Kitamura [Tôhoku Math. J. **49** (1943), 305–307; MR **8**, 517; Theorem 2]. The authors' suppositions are the following. (I) $f(x)$ is continuous and strictly increasing in $[a, b]$, where $a$, $b$ are consecutive roots of $f(x) = x$. (II) $F(x, y, z)$ is continuous in a convex region $R$ and has there continuous derivatives $F_y \neq 0$, $F_z \neq 0$. (III) $F(x, y, z) = 0$ represents in $R$ a simple connected surface. (IV) $(a, b)$ is contained in a set for each point of which there exist such $y$ and $z$, that $(x, y, z) \in R$ and $F(x, y, z) = 0$. (V) The projection on the $z$-axis of the intersection-curve of surface $F(x, y, z) = 0$ with plane $x = x_0$ is identical with the projection on the $y$-axis of the intersection-curve of the same surface with plane $x = f(x_0)$. Under these hypotheses the functional equation in question has an infinite number of solutions in $(a, b)$, but it is shown by counter-examples that the theorem loses validity if any of the hypotheses (III), (IV) and (V) are omitted [cf. Kitamura, loc. cit.]. If $F(b, d, d) = 0$ and $|F_y(x, y, z)/F_z(x, y, z)| \geqq 1$ in a neighborhood of $(b, d, d)$, then there exists at most one solution $\varphi(x)$ of the functional equation, continuous in $(a, b]$, which satisfies $\varphi(b) = d$; and there is a similar situation for $a$ instead of $b$.—The authors give also a construction for this unique solution. Nevertheless they don't regard their results as quite satisfactory and, the reviewer was informed, the Kraków mathematicians are continuing to investigate such functional equations and have found further results since.

*J. Aczél* (Debrecen)

## SEQUENCES, SERIES, SUMMABILITY

See also 747, 890.

851:

**Obrešlkov, N. Sur la sommation absolue des séries divergentes par les moyennes arithmétiques.** Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 2, 39–61 (1959). (Bulgarian. Russian and French summaries)

Résume de l'auteur: "Dans ce travail nous démontrons les théorèmes suivants. 1. Désignons par $s_n^k$ les moyennes arithmétiques d'ordre $k$ de la série $u_0+u_1+u_2+\cdots$ et supposons que la série $\sum_{n=1}^{\infty}|t_n-t_{n+1}|$, $t_n=n^{-1}\sum_{\nu=1}^{n}\nu u_\nu$ est convergente. Pour $k$ entier la série $\sum_{n=1}^{\infty}|s_n^k-s_{n-1}^k-u_n|$ sera aussi convergente. 2. Soit $\{\lambda_n\}_1^\infty$ une suite de nombres réels qui satisfont aux conditions $0<\lambda_1<\lambda_2<\lambda_3<\cdots$, $\lambda_n\to\infty$. La condition nécessaire et suffisante pour que de la sommabilité $|R,\lambda,1|$ de la série $\sum_{n=1}^{\infty}a_n$ il découle la convergence absolue de la même série consiste en ce que la suite $\{t_n\}_1^\infty$, où $t_n=(1/\lambda_{n+1})\sum_{\nu=1}^{n}\lambda_\nu a_\nu$, soit de variation bornée. 3. Soit $\{n_\mu\}_1^\infty$ et $\{m_\mu\}_1^\infty$ deux suites de nombres positifs et entiers, tels que $n_\mu<m_\mu\leqq n_{\mu+1}$, $\mu=1, 2, 3,\cdots$, $\lambda_{m_\mu}/\lambda_{n_\mu}>q>1, \mu=1,2,\cdots$. Supposons que pour les membres $u_n$ de la série $u_1+u_2+u_3+\cdots$ on a $u_n=0$ pour $n_\mu<n<m_\mu$, $\mu=1, 2, \cdots$, et que cette série est sommable $|R,\lambda,1|$. Alors la série $\sum_{\mu=1}^{\infty}|\sum_{n=m_\mu}^{n_{\mu+1}}u_n|$ sera convergente. On démontre aussi des résultats semblables pour la sommabilité absolue $|C,k|$ des intégrales."

852:

**Wells, J. H. Concerning the Hausdorff inclusion problem.** Duke Math. J. **26** (1959), 629–645.

The author considers Hausdorff means generated by sequences $a_n=\int_0^1 x^n d\varphi(x)$, where $\varphi(x)$ is a real-valued function defined on $-\infty<x<+\infty$ and satisfying

(1) $\varphi(x)=0$ $(x\leqq 0)$, $\varphi(x)=\varphi(1)$ $(x>1)$,

(2) $\varphi(x)$ is quasicontinuous on $[0, 1]$ and

(3) $\varphi(x)=\frac{1}{2}(\varphi(x+)+\varphi(x-))$ $(0<x<1)$.

Necessary and sufficient conditions for a sequence $a_n$ to allow of a representation $a_n=\int_0^1 x^n d\varphi(x)$ are given. Using a generalisation of Stieltjes integrals, an inclusion theorem for those means is given, also necessary and sufficient conditions for a regular Hausdorff mean to include $(C,r)$ $(r>0)$.

*K. Endl* (Columbus, Ohio)

853:

**Mayer-Kalkschmidt, J. On conditional inclusion of matrix methods.** Proc. Amer. Math. Soc. **10** (1959), 193–198.

Der Verf. verallgemeinert Vergleichssätze von Jurkat und dem Ref. [Math. Z. **56** (1952), 152–178; MR **14**, 158; erratum, **15**, 1139] auf den Fall, wo Summierbarkeit mit verschiedener "Geschwindigkeit" verglichen wird. Es wird u.a. gezeigt, dass $\sum_{\nu\leq n}a_{n,\nu}s_\nu=O(\Omega(n))$, $0<\Omega(n)\nearrow$, $a_{n,\nu}\neq 0$ für $\nu\leqq n$, die Beziehung $\sum_{\nu\leq n}b_{n,\nu}s_\nu=O(1)$ nach sich zieht, falls $\sum_{\nu\leq n}\Omega(\nu)|\Delta_\nu(b_{n,\nu}/a_{n,\nu})|=O(1)$ und

$$(*)\qquad \Big|\sum_{\nu\leq n}a_{m,\nu}s_\nu\Big|\leqq K\Big|\sum_{\nu\leq n'}a_{n',\nu}s_\nu\Big|$$

gilt für jedes Paar $n<m$ und ein $n'<n$. Unter Verwendung der Inversen werden von der Voraussetzung (*) freie Sätze dieser Art bewiesen. {In Theorem 2 und der ersten Hälfte von Theorem 3 wird $a_{n,\nu}\to 0$ für $n\to\infty$, $\nu$ fest nicht benötigt; in Corollary 3 ist $O$ durch $o$ zu ersetzen.} Als Anwendung wird der Schluss von Cesàro nach Euler ($s_0+\cdots+s_n=(n+1)c+o(n^{1/2})$ impliziert Euler) abgeleitet.

*A. Peyerimhoff* (Marburg)

854:

**Agnew, Ralph Palmer. Relations among the Lototsky, Borel and other methods for evaluation of series.** Michigan Math. J. **6** (1959), 363–371.

The series $\sum u_n$ is summable Lototsky to $S$ if the series $\sum U_n$ converges to $S$ where

$$U_n=\sum_{k=1}^{n}\frac{p_{n,k}}{(n+1)}u_k \quad (n=1,2,\cdots)$$

and $p_{n,k}$ is the coefficient of $x^k$ in the polynomial $x(x+1)(x+2)\cdots(x+n-1)$. The author establishes a number of relations between this method and Borel's methods and the Abel method of summability. He shows that the Lototsky method is not Fourier-effective and that the generalized Borel integral method is equivalent to the Lototsky method followed by the Abel method. Other inclusion results are given and there is an example of a series which is evaluable by both Borel's methods but not by the Lototsky method. It is also shown that the generalized Abel method and the Lototsky method followed by the generalized Abel method are consistent.

*H. G. Eggleston* (Cambridge, England)

855:

**Yurimyaè, È. I. [Jürimäe, E.] Einige Fragen über verallgemeinerte Matrixverfahren, co-reguläre und co-null-Verfahren.** Eesti NSV Tead. Akad. Toimetised. Tehn. Füüs.-Mat. Tead. Seer. **8** (1959), 115–121. (Russian. Estonian and German summaries)

Der Verf. betrachtet Transformationen $y_n=\sum_k A_{nk}x_k$, wo $x_k$, $y_n$ Elemente von Banachräumen $X$, $Y$ und die $A_{nk}$ lineare Operatoren sind (Transformationen dieser Art wurden behandelt von A. Robinson, Proc. London Math. Soc. (2) **52** (1950), 132–160 [MR **12**, 253], H. Melvin-Melvin, ibid. **53** (1951), 83–108 [MR **13**, 45], K. Zeller, Math. Z. **56** (1952), 18–20 [MR **14**, 158]). In Ahnlehnung an A. Zeller [Math. Z. **53** (1951), 463–487; MR **12**, 604] und A. Wilansky [Trans. Amer. Math. Soc. **67** (1949), 59–68; MR **11**, 243] wird im die Wirkfelder eine $FK$-Topologie eingeführt, und es werden Begriffe co-regulär und co-null erklärt (ist eine Transformation konvergenzerhaltend, so ist sie co-null, wenn die Folge $e_r^*(x)=\{0, 0, \cdots, 0, x, x, \cdots\}$ mit $r$-Nullen am Anfang für jedes $x\in X$ im Wirkfeld schwach gegen Null konvergiert, sonst coregulär). Als Anwendung wird u.a. gezeigt, dass ein reguläres Verfahren, das eine beschränkte divergente Folge limitiert, auch eine unbeschränkte Folge limitiert.

*A. Peyerimhoff* (Marburg)

856:

**Smart, D. R. On $o$-Tauberian theorems.** Quart. J. Math. Oxford Ser. (2) **10** (1959), 140–144.

Verfasser betrachtet permanente Matrixverfahren $H$ und $o$-Bedingungen (i) $a_n=o(k_n^{-1})$ (wo $k_n>1$) für die Reihenglieder $a_n=A_n-A_{n-1}$ einer ($H$-limitierbaren) Folge $(A_n)$. Er zeigt: Folgt aus (i) und $H$-Limitierbarkeit die Beschränktheit von $(A_n)$, so sogar die Konvergenz. Zum Beweis führt er in den Raum der zugelassenen Folgen die

Norm $\|A\| = \sup k_n|a_n| + \sup |(HA)_t|$ ein und zeigt, daß in dem entstehenden $B$-Raum die abbrechenden Folgen dicht liegen. Das Grundmengenprinzip (oder der Übergang zur Norm $\sup k_n|a_n| + \sup |A_n|$) liefert die Behauptung.—{Die vom Verf. noch angeführte Bedingung über die $RF$-Form ist—wie er vermutet—überflüssig. Ähnliche Ergebnisse findet man bei Referent, Dissertation [Tübingen, 1950] und Math. Z. **53** (1951), 463–487 [MR **12**, 604].}

*K. Zeller* (Tübingen)

857:

**Kuttner, B. On bounded bilinear forms, and the summability of a Cauchy product series.** Proc. London Math. Soc. (3) **9** (1959), 556–574.

In part I of the paper, some properties of a bilinear form $A$ are considered, where $A = \sum a_{p,q}x_p y_q$, the $a$'s, $x$'s and $y$'s being complex numbers, and the sum being taken over values of $p$ and $q$ from 0 to $\infty$. The set of sequence pairs $(x, y)$ for which $|x_p| \leq 1$, $|y_q| \leq 1$ $(p, q = 0, 1, 2, \cdots)$ is denoted by $S$, and the set $(x, y)$ for which $x_p \to 0$ as $p \to \infty$, $y_q \to 0$ as $q \to \infty$ is denoted by $S^*$. The following known results are summarized as theorem A. For a given bilinear form $A$, the following assertions are equivalent: (a) there is a constant $K$ such that, for every $(x, y)$ of $S$ and all $P, Q$,

$$\Big|\sum_{p=0}^{P}\sum_{q=0}^{Q} a_{p,q}x_p y_q\Big| \leq K;$$

(b) the double sum $A$ converges in the Pringsheim sense for every $(x, y)$ of $S$; (b*) the double sum $A$ converges in the Pringsheim sense for every $(x, y)$ of $S^*$; (c) the double sum $A$ converges by rows for every $(x, y)$ of $S$; (c*) the double sum $A$ converges by rows for every $(x, y)$ of $S^*$. The author then considers the sum by diagonals $\lim_{N\to\infty}\sum_{p+q\leq N} a_{p,q}x_p y_q$ if it exists, or, equivalently, $\sum_{n=0}^{\infty}\lambda_n$, where $\lambda_n = \lambda_n(x, y) = \sum_{p=0}^{n} a_{p,n-p}x_p y_{n-p}$; he is then led to the problem of finding what relations of implication (if any) hold among the following assertions themselves, or between them and the assertions of theorem A. (d) There is a $K$ such that, for every $(x, y)$ of $S$ and all $N$,

$$\Big|\sum_{p+q\leq N} a_{p,q}x_p y_q\Big| = \Big|\sum_{n=0}^{N}\lambda_n\Big| \leq K;$$

(e) $\sum_{n=0}^{\infty}\lambda_n$ converges for every $(x, y)$ of $S$; (e*) $\sum_{n=0}^{\infty}\lambda_n$ converges for every $(x, y)$ of $S^*$. The solution of this problem is given by theorem 1: The assertions (d) and (e*) are equivalent. Also, if (A) denotes the assertion $\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}|a_{p,q}| < \infty$, then

(1) $$(A) \to (e) \to (d) \to (a).$$

The converse of each one of the implications in (1) is false. Theorem 2: The converse of each one of the implications in (1) remains false if the matrix $a_{p,q}$ has the form $a_{p+q}$. Theorem 3: If (a) holds, then there is a $K$ such that, for every $(x, y)$ of $S$ and all $N$, $|\sum_{n=0}^{N}\lambda_n| \leq K \log(N+2)$.

In part II of the paper, the following summability problem is considered. Let $\mathfrak{C}$ denote the class of sequences $\{u_m\}$ formed by taking the partial sums of the Cauchy product series of any two convergent series; for any member of $\mathfrak{C}$, $s, t$ are used to denote the sums of these two series. The problem is to find a set of necessary and sufficient conditions for the sequence-to-sequence transformation $T$ given by $v_n = \sum_{m=0}^{\infty}\beta_{n,m}u_m$ to sum every member of $\mathfrak{C}$ to $st$. The following results are proved. Theorem 4: In order that $\sum_{m=0}^{\infty}\beta_m u_m$ should converge for every sequence $\{u_m\}$ of $\mathfrak{C}$, it is necessary and sufficient that $\beta_m \to 0$ as $m \to \infty$, and that the bilinear form

$$\sum(\beta_{p+q} - \beta_{p+q+1})x_p y_q$$

should be diagonally bounded in $S$. If these conditions are satisfied, then, for every $\{u_m\}$ of $\mathfrak{C}$,

$$\sum_{m=0}^{\infty}\beta_m u_m = \sum_{m=0}^{\infty} d_m \sum_{r=0}^{\infty}\beta_{m+r}s_r,$$

where $s_m = c_0 + c_1 + \cdots + c_m \to s$, $t_m = d_0 + d_1 + \cdots + d_m \to t$, as $m \to \infty$. The main result of part II is theorem 5: In order that the transformation $T$ defined above by $v_n$ should be such that every member of $\mathfrak{C}$ is $T$-summable to $st$, the following set of conditions is necessary and sufficient: (i) $T$ is regular, (ii) for every fixed $n$, the bilinear form $A_n = \sum(\beta_{n,p+q} - \beta_{n,p+q+1})x_p y_q$ is diagonally bounded in $S$, (iii) the forms $A_n$ are bounded in $S$, uniformly in $n$.

*R. G. Cooke* (London)

## APPROXIMATIONS AND EXPANSIONS

See also 900, B1101, B1102.

858:

**Hsu, L. C. The polynomial approximation of continuous functions defined on $(-\infty, \infty)$.** Czechoslovak Math. J. **9** (84) (1959), 574–578. (Russian summary)

If $0 < \theta < \frac{1}{2}$ and $f(x)$ is bounded and continuous on $(-\infty, +\infty)$, then

$$\lim_{n\to\infty}(n/\pi)^{1/2}\int_{-1}^{+1} f(n^\theta t)[1-(t-n^{-\theta}x)^2]^n dt = f(x),$$

uniformly on each finite interval. For $\theta = 0$, this polynomial reduces to Landau's singular integral.

*G. G. Lorentz* (Syracuse, N.Y.)

859:

**Ionescu, D. V. Formules de quadrature à nœuds extérieurs.** Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **9** (1958), 45–134. (Romanian. Russian and French summaries)

Quadrature formulas with exterior nodes, as well as the remainders, are studied here in detail. The author develops a number of formulas, with applications and examples, and relations to some known formulas.

*E. Frank* (Chicago, Ill.)

860:

**Stroud, A. H. Numerical integration formulas of degree two.** Math. Comput. **14** (1960), 21–26.

The author gives a method of devising quadrature formulas of the form

$$\int_R f(x)W(x)dx \doteq \sum_i a_i f(x_i)$$

for arbitrary regions $R$ in $n$-space. The method establishes conditions on $W(x)$ under which a formula involving $n+1$ points $x_i$ may be established so as to be exact for polynomial functions $f(x)$ of at most second degree in $n$ variables. To achieve such a formula it is necessary to be able to integrate $W(x)$ and all monomials and binomials times $W(x)$ over $R$. For this degree the author shows that

$n+1$ points are also necessary and this is the first nontrivial result of this kind for $n$-space. The development draws from the theory of quadratic forms and functions.

It is also shown that an analogous method produces a third degree formula based on $2n$ points for centrally symmetrical regions $R$. It is noted that the weight function $W(x)$ need not be positive valued, although in the third degree application it must be symmetrical.

*P. C. Hammer* (Madison, Wis.)

861:

**Krylov, V. I. Mechanical quadratures with equal coefficients for the integrals**

$$\int_0^\infty e^{-x}f(x)dx \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-x^2}f(x)dx.$$

Dokl. Akad. Nauk BSSR **2** (1958), 187–192. (Russian)

H. Salzer's computations [J. Math. Phys. **34** (1955), 103–112; MR **17**, 538] of the abscissas for the quadrature formulas of the types

$$\int_0^\infty e^{-x}f(x)dx \approx \frac{1}{n}\sum_{k=1}^{n} f(x_k), \quad \int_{-\infty}^{\infty} e^{-x^2}f(x)dx \approx \frac{\sqrt{\pi}}{n}\sum_{k=1}^{n} f(x_k)$$

are examined to show that the complex $x_k$ always exist for both formulas for any $n>10$. Salzer found that among the $x_k$ there are complex values for $3\leq n\leq 10$ for the first formulas and for $4\leq n\leq 10$ for the second.

The proof is based on the examination of an arbitrary quadrature formula with $n$ abscissas $x_k$, $k=1, 2, \cdots, n$ $(x_k<x_{k+1})$ and positive coefficients $p_k$

$$\int_a^b \rho(x)f(x)dx \approx \sum_{k=1}^{n} p_k f(x_k), \tag{1}$$

and the quadrature formula with $m$ abscissas $\xi_j$, $j=1, 2, \cdots, m$ $(\xi_j<\xi_{j+1})$,

$$\int_a^b \rho(x)f(x)dx \approx \sum_{j=1}^{m} A_j f(\xi_j),$$

having the highest degree of precision $2m-1$. The $\xi_j$ are the roots of the polynomial of degree $m$, orthogonal on $[a, b]$ with the weight $\rho(x)$ to any polynomial of lower degree. The $A_j$ are computed according to established known rules.

The proof is an extension of the results by S. N. Bernšteĭn: If formula (1) is correct for all polynomials of degree $2m-1$ $(m<n)$ then $x_n>\xi_m$; if formula (1) is correct for all polynomials of degree $2m-1$ $(m<n)$ then $p_n<A_m$. Three papers by S. N. Bernšteĭn are referred to. They appear in his *Sobranie sočineniĭ*, Tom II [Izdat. Akad. Nauk SSSR. Moskow, 1954; MR **16**, 433; pp. 198–204, 236–242; the papers appeared as C. R. Acad. Sci. Paris **203** (1936), 1305–1306; Dokl. Akad. Nauk SSSR **14** (1937), 323–327; J. Math. Pures Appl. **17** (1938), 179–186].

*A. Bakst* (Flushing, N.Y.)

862:

**Agnew, Ralph Palmer. Convergence of orthogonal series.** Duke Math. J. **27** (1960), 127–131.

Given a sequence of complex numbers $a_n$ such that $\sum_0^\infty |a_n|^2=\infty$ and a real one-dimensional interval $I$, a set $\phi_0(x)$, $\phi_1(x)$, $\cdots$, $\phi_n(x)$, $\cdots$ orthonormal over $I$ is constructed for which the series

$$a_0\phi_0(x)+a_1\phi_1(x)+\cdots+a_n\phi_n(x)+\cdots = 0$$

converges to the sum zero for each $x$ in $I$. The functions $\phi_n(x)$ are not uniformly bounded in $I$, but they are continuous over $I$, and the examples in which the orthonormal set is complete can be obtained.

*E. Kogbetliantz* (New York)

863:

**Ciesielski, Z.; Musielak, J. On absolute convergence of Haar series.** Colloq. Math. **7** (1959), 61–65.

By a straightforward calculation theorems analogous to those given by Bernstein and Zygmund for trigonometric series are proved. If $\chi_n(t)$ are the Haar functions,

$$f \sim \sum a_n\chi_n(t),$$

$$\omega(h) = \sup_{\delta,t} |f(t+\delta)-f(t)| \quad (|\delta| \leq t \leq 1-|\delta|, |\delta| < h),$$

$$\omega_r(h) = \sup_{\delta} \left[\int_0^1 |f(t+\delta)-f(t)|^r dt\right]^{1/r} \quad (|\delta| \leq h, r \geq 1),$$

$\beta\geq 0$, $\gamma$, $\lambda>0$, $\mu=\max(\gamma, 1)$, then $\sum_{n=1}^\infty \omega^\gamma(1/n)n^{\beta+(\gamma-\lambda)/2-1}<\infty$ implies $\sum_{n=1}^\infty n^\beta|a_n|^\gamma|\chi_n(t)|^\lambda$ converges uniformly and $\sum_{n=1}^\infty \omega_\mu^\gamma(1/n)n^{\beta-\gamma/2}<\infty$ implies $\sum_{n=1}^\infty n^\beta|a_n|^\gamma<\infty$. Similar theorems hold (also for the Rademacher functions) if $f$ is of bounded $p$th variation $(p\geq 1)$, and corollaries imply uniform convergence of $\sum_{n=1}^\infty |a_n\chi_n(t)|$, or $\sum_{n=1}^\infty |a_n|<\infty$.

*G. Goes* (Evanston, Ill.)

864:

**Lorch, Lee. The Lebesgue constants for Jacobi series. I.** Proc. Amer. Math. Soc. **10** (1959), 756–761.

The author proves that the Lebesgue constants $L_n(\alpha, \beta)$ for Jacobi series are

$$A_{\alpha,\beta}n^{\alpha+\frac{1}{2}}+O(n^{\alpha-\frac{1}{2}})+O(n^{\alpha-\beta-1})+O(1) \tag{1}$$

for $\alpha>-\frac{1}{2}$, $\alpha\neq\frac{1}{2}$ and $\beta\neq-\frac{1}{2}$, where

$$A_{\alpha,\beta} = \frac{2}{\pi^{\frac{1}{2}}}\frac{\Gamma(\frac{1}{2}\alpha+\frac{1}{4})\Gamma(\frac{1}{2}\beta+\frac{3}{4})}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}[\alpha+\beta]+1)}.$$

If $\alpha=\frac{1}{2}$, then the second term of (1) must be replaced by $O(\log n)$, and if $\beta=-\frac{1}{2}$, then the third term of (1) must be replaced by $O(n^{\alpha-\frac{1}{2}}\log n)$. This result generalizes H. Rau's result [J. Reine Angew. Math. **161** (1929), 237–254]. For the proof the author uses the representation

$$L_n(\alpha, \beta) = \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \times \int_0^\pi (\sin\tfrac{1}{2}\theta)^{2\alpha+1}(\cos\tfrac{1}{2}\theta)^{2\beta+1}|P_n^{(\alpha+1,\beta)}(\cos\theta)|d\theta.$$

Moreover, the author gives the deep estimation of

$$L_n(\alpha, \beta, \mu, \lambda) = \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \times \int_0^\pi (\sin\tfrac{1}{2}\theta)^{\mu}(\cos\tfrac{1}{2}\theta)^{\lambda}|P_n^{(\alpha+1,\beta)}(\cos\theta)|d\theta.$$

*S. Izumi* (Sapporo)

865:

**Dzyadyk, V. K. On a problem of S. M. Nikol'skiĭ in a complex region.** Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 697–736. (Russian)

This is a detailed exposition of the author's paper in Dokl. Akad. Nauk SSSR **127** (1959), 505–508 [MR **21** #6444].

*G. G. Lorentz* (Syracuse, N.Y.)

866:

**Berman, D. L. On the impossibility of constructing a linear polynomial operator giving an approximation of best order.** Uspehi Mat. Nauk **14** (1959), no. 4 (88), 141–142. (Russian)

Using a generalization of an identity of Marcinkiewicz [D. L. Berman, Dokl. Akad. Nauk SSSR **85** (1952), 13–16; **120** (1958), 1175–1177; MR **14**, 57; **20** #5387] the author gives an elegant proof of the following fact. There does not exist a sequence of continuous linear operators $V_n(f)$ mapping $C[-1, +1]$ into itself such that, for each $f \in C$, (1) $V_n(f)$ is an algebraic polynomial of degree $n$, (2) $\|f - V_n(f)\| \leqq O(E_n(f))$. {Actually, condition (2) may be weakened to $\|f - V_n(f)\| \to 0$, as may be derived from Natanson, *Konstruktive Funktionentheorie* [Akademie-Verlag, Berlin, 1955; MR **16**, 1100; Appendix].}

*G. G. Lorentz* (Syracuse, N.Y.)

867:

**Gončar, A. A. Inverse theorems on best approximations on closed sets.** Dokl. Akad. Nauk SSSR **128** (1959), 25–28. (Russian)

It is known [Gončar, same Dokl. **100** (1955), 205–208; MR **16**, 803] that there does not exist a relation between the degree of approximation of a function $f$ by rational functions of order $n$ and its continuity properties on a closed linear set $F$. It is shown here that this unfavorable situation changes if $F$ has positive Lebesgue measure and if one considers the continuity properties of $f$ "in the interior of $F$". Let $f \in L(k+\alpha, P)$, where $k = 0, 1, \cdots$, $0 < \alpha \leqq 1$ and $P$ is a perfect set, signify that the restriction of $f$ to $P$ has $k$th derivative $f^{(k)}$ which satisfies a Lipschitz condition of order $\alpha$ on $P$. And $f \in L(k+\alpha, F-)$, where $F$ is a closed set, means that for each $\varepsilon > 0$ a perfect set $P_\varepsilon \subset F$ exists with meas $(F - P_\varepsilon) < \varepsilon$ and $f \in L(k+\alpha, P_\varepsilon)$. Let $R_n(f, F)$ be the degree of approximation of $f$ on $F$ by rational functions of order $\leqq n$ (i.e., by quotients of polynomials of degrees $\leqq n$), let $R_n^*(f, F) = \inf R_n(f, P_\varepsilon)$, where the infimum is taken for all perfect sets $P_\varepsilon \subset F$ such that meas $(F - P_\varepsilon) < \varepsilon$ and that $f$ is continuous on $P_\varepsilon$. Let $E_n^*(f, F)$ be the same quantity with respect to the approximation by polynomials of degree $\leqq n$. The main results of the author are: If for some $A > 0$ and each $\varepsilon > 0$, $R_n^*(f, F) = O(n^{-A-\delta})$ with $\delta = \delta(\varepsilon) > 0$, then $f \in L(A, F-)$. If $f \in L(A, F-)$, $A > 0$, then for each $\varepsilon > 0$, $E_n^*(f, F) = O(n^{-A})$. The author also discusses the quasi-analytic class $R(F)$ of functions $f$ on $F$ defined by $\lim_{n\to\infty} [R_n^*(f, F)]^{1/n} = 0$ for each $\varepsilon > 0$.

*G. G. Lorentz* (Syracuse, N.Y.)

868:

**Pressman, A. Ya.; Berlyand, O. S. On obtaining asymptotic expressions for a certain class of functions.** Dokl. Akad. Nauk SSSR **126** (1959), 508–510. (Russian)

Let

$$\delta(\nu, t) = \frac{\nu^{\nu+1}}{\Gamma(\nu+1)} t e^{-\nu t},$$

and note that $\delta(\nu, t) \to \delta(t-1)$ as $\nu \to \infty$. It is formally shown that if $a < 1 < b$, then

$$\int_a^b \varphi(x, t)\delta(\nu, t)dt \sim \varphi(x, 1) - \frac{1}{12\nu}\frac{d^4}{dt^4}[\varphi(x, t)t^2(1 - t + \ln t)]_{t=1}.$$

Special cases are discussed. For example, if $\varphi = \exp(-x^2t^2/2)$, $a = 0$ and $b = \infty$, then the formula above yields an asymptotic representation for the parabolic cylinder function $D_{-(\nu+1)}(\nu/x)$.

*N. D. Kazarinoff* (Moscow)

869:

**Schoenfeld, Lowell. The asymptotic expansion of**

$$\int_0^1 \frac{x^2\tau^2 A(\tau) + x\tau B(\tau) + C(\tau)}{\lambda x^2\tau^2 + x\tau D(\tau) + E(\tau)}\, d\tau.$$

J. Math. Phys. **38** (1959/60), 193–202.

Let $I(x)$ be the integral given in the title. Assuming that $A, \cdots, E$ are continuous functions of $\tau$ on $[0, 1]$, independent of $\lambda$ and $x$, assuming further that for some $\varepsilon > 0$,

$$A(\tau) = a_0 + a_1\tau + O(\tau^{1+\varepsilon}) \quad \text{as } \tau \to +\infty,$$

and similar asymptotic forms are available for $B, \cdots, E$, and assuming lastly that $d_0, e_0 > 0$; the author proves

$$I(x) = K_0 + K_1\frac{\log x}{x} + K_2\frac{1}{x} + K_3\frac{\log x}{x^2} + O\left(\frac{\log(\lambda+2)}{\lambda x^2}\right),$$

where $x \to +\infty$, $\lambda \geqq \delta > 0$, the $K_r$ depend on $\lambda$ (and are given explicitly in the paper), and the constant implied in the $O$-symbol does not depend on $\tau$, $\lambda$, or $x$. The case of identically vanishing $C$ and $E$ is somewhat different and is treated separately; and some other modifications of the principal result are also discussed.

*A. Erdélyi* (Pasadena, Calif.)

## FOURIER ANALYSIS

See also 886.

870:

**Dzyadyk, V. K. On the best trigonometric approximation in the $L$ metric of some functions.** Dokl. Akad. Nauk SSSR **129** (1959), 19–22. (Russian)

Several theorems listed in this paper belong to one of two types. Some of them assert that a trigonometric polynomial $T_n(t)$ of degree $\leqq n$ can coincide with a function $K(t)$ of a certain class in at most $m = m(n)$ points in $0 \leqq t < 2\pi$. This is the case, for example, if $m = 2n+1$, and $K(t)$ has an absolutely monotone derivative for $-\infty < t < 2\pi$. Theorems of the second type postulate the situation asserted by theorems of the first type and give information about polynomials of best approximation in the $L$-norm. Thus, (a) if $\varphi(t)$ is continuous and integrable on $(0, 2\pi)$, if finite or infinite limits $\varphi(0+)$, $\varphi(2\pi-)$ exist, and if each equation (*) $\varphi(t) - T_n(t) = 0$ has at most $2n+2$ roots in $(0, 2\pi)$, then there exists a $\xi_0$, $0 < \xi_0 \leqq \pi/(n+1)$ such that the trigonometric polynomial of degree $\leqq n$ which interpolates $\varphi(t)$ at the points $\xi_0 + t_k$, $t_k = k\pi/(n+1)$, $k = 0, 1, \cdots, 2n$, is the polynomial of best approximation of $\varphi$ in the $L$-norm. Methods of determining $\xi_0$ are given. For example, if $a_k$, $b_k$ are the Fourier coefficients of $\varphi$, then

$$\sum_{j=0}^{\infty} \{a_{(n+1)(2j+1)} \cos(n+1)(2j+1)\xi_0 + b_{(n+1)(2j+1)} \sin(n+1)(2j+1)\xi_0\} = 0.$$

(b) The same conclusion applies to the periodic integral $\varphi(t) = \psi_s(t)$ of order $s = 1, 2, \cdots$ of a continuous and integrable in $(0, 2\pi]$ function $\psi(t)$, provided equation (*) has again at most $2n+2$ roots in $(0, 2\pi)$. The author adds that his results furnish a solution of the problem of

best approximation in the class of functions which have a bounded fractional derivative of order $s$, $0<s<+\infty$, proposed by Favard [Bull. Sci. Math. (2) **61** (1937), 209–224, 243–256]. *G. G. Lorentz* (Syracuse, N.Y.)

871:

**Ibragimov, I. I. Extremal problems in a class of trigonometric polynomials.** Izv. Akad. Nauk Azerbaĭdžan. SSR. Ser. Fiz.-Teh. Him. Nauk **1958**, no. 2, 3–17. (Russian. Azerbaijani summary)

The author considers trigonometric polynomials $T$, in $k$ variables, of order $(n_1, \cdots, n_k)$, and solves some extremal problems for functionals $I$ obtained by taking the mean value of $KT$ over a period cube. His fundamental result is that if $K$ has Fourier coefficients $b_{\nu_1,\cdots,\nu_k}$ then

$$|I| \leq (2\pi)^{-k/p}\|T\|_p\Big(\sum_{-n_1}^{n_1}\cdots\sum_{-n_k}^{n_k}|b_{\nu_1,\cdots,\nu_k}|^p\Big)^{1/p}$$

if $1\leq p\leq 2$. Several special cases are discussed. The author deduces from one of them that if $1\leq p\leq 2$ and $p'\geq p\geq 1$ then

$$\|T\|_{p'} \leq \Big\{\prod_{j=1}^{k}(n_j/\pi)\Big\}^{(1/p)-(1/p')}\|T\|_p,$$

sharpening an inequality of Nikol'skiĭ's [Trudy Mat. Inst. Steklov. **38** (1951), 244–278; MR **14**, 32]. Applications to algebraic polynomials. *R. P. Boas, Jr.* (Evanston, Ill.)

872:

**Sunouchi, Gen-ichiro; Watari, Chinami. On the determination of the class of saturation in the theory of approximation of functions. II.** Tôhoku Math. J. (2) **11** (1959), 480–488.

Nouvelle méthode de recherche des classes de saturation de divers procédés classiques de sommation des séries de Fourier des fonctions périodiques appartenant à certains espaces fonctionnels.

Soit $f(x)\in L^p$ une fonction de période $2\pi$ ($p\geq 1$, $L^\infty=C$), et sa série de Fourier

$$f(x)\sim\frac{a_0}{2}+\sum_1^\infty(a_k\cos kx+b_k\sin kx)=\sum_0^\infty A_k(x).$$

On applique une méthode de sommation $G$ définie par une fonction $g_k(n)$ ($k=0, 1, 2, \cdots$; $g_0(n)=1$) et on pose $P_n(x)=\sum_0^\infty g_k(n)A_k(x)$; alors s'il existe une fonction décroissante au sens large $\varphi(n)$ et une classe $K$ de fonctions telles que

$$\|f-P_n\| = o[\varphi(n)] \to f(x) = \text{const},$$
$$\|f-P_n\| = O[\varphi(n)] \leftrightarrow f(x)\in K,$$

le procédé de sommation est saturé avec l'ordre $\varphi(n)$, et sa classe de saturation est $K$.

La méthode de recherche est basée sur le résultat suivant: s'il existe une fonction positive $\psi(k)$ telle que

$$\lim_{n\to\infty}\frac{1-g_k(n)}{\varphi(n)}=\psi(k)\quad(k=1, 2, \cdots),$$

alors on a, quel que soit $N$,

$$\Big\|\sum_{k=1}^{N}f(k)\Big(1-\frac{k}{N+1}\Big)A_k(n)\Big\| = O(1)$$

et on en déduit immédiatement que $\sum_1^\infty\psi(k)A_k(x)$ est la série de Fourier d'une fonction de $L^p$, pour $p>1$, ou la série de Fourier-Stieltjes d'une fonction à variation bornée pour $p=1$.

Application aux procédés de Fejér, Cesàro, Abel-Poisson, Riesz, Gauss-Weierstrass, Bernstein-Rogosinski, de la Vallée-Poussin, Jackson. *R. Campbell* (Caen)

873:

**Varshney, O. P. On a sequence of Fourier coefficients.** Proc. Amer. Math. Soc. **10** (1959), 790–795.

Let $\sum a_k$ be a given series and $s_n$ be its $n$th partial sum. If the sequence $t_n=(\log n)^{-1}\sum_{k=0}^{n}(k+1)^{-1}s_{n-k}$ converges to a number $t$, then the series $\sum a_k$ is said to be summable $(N, 1/(n+1))$ to $t$. The author proves the following theorem. Let $f(t)$ be an integrable function and $B_n(x)$ be the $n$th term of the conjugate Fourier series of $f$. If

$$(1)\qquad \int_0^t|f(x+t)-f(x-t)-l|\,dt = o(t/|\log t|)$$

as $t\to 0$ for a fixed $x$, then the sequence $\{nB_n(x)\}$ is summable $(N, 1/(n+1))\cdot(C, 1)$ to $l$ where the summation is the iteration of $(N, 1/(n+1))$ and $(C, 1)$. Combining this theorem with Iyengar's Tauberian theorem [Proc. Indian Acad. Sci. Sect. A **18** (1943), 81–87; MR **5**, 65], the author gets the theorem: If (1) holds and the Fourier coefficients of $f$ are of order $O(n^{-\delta})$ $(0<\delta<1)$, then the sequence $\{nB_n(x)\}$ is summable $(C, 1)$. He derives also a known convergence criterion of conjugate Fourier series from the first theorem. *S. Izumi* (Sapporo)

874:

**Sorokina, N. G. On a theorem of N. N. Bogolyubov.** Ukrain. Mat. Ž. **11** (1959), 220–222. (Russian)

A new discussion is given of a theorem by Bogolyubov on the analyticity properties of the Fourier transforms of certain functions vanishing in the interior or exterior of the light-cone in 4-dimensional Minkowski space. This theorem is of importance in the discussion of dispersion relations in quantized field theories. The present paper may be considered as an extension of that of F. J. Dyson [Phys. Rev. (2) **110** (1958), 1460–1464; MR **20** #1537]. *E. L. Hill* (Minneapolis, Minn.)

875:

**Devinatz, Allen. On the extensions of positive definite functions.** Acta Math. **102** (1959), 109–134.

The paper is concerned with the following problem. Let $G$ be an Abelian locally compact group, $Q$ a symmetric neighborhood of the identity, and $f$ a continuous function defined on $2Q$ and positive definite on $Q$: for any set $\{x_k\}_1^n\subset Q$, and complex numbers $\{\zeta_k\}_1^n$,

$$(*)\qquad \sum_{j=1}^{n}\sum_{k=1}^{n}\zeta_j\bar{\zeta}_k f(x_j-x_k)\geq 0.$$

Is it possible to extend $f(x)$ to all of $G$ so as to retain property (*)? In the case $G=$ the real line, the problem was stated and answered in the affirmative by M. Kreĭn, and later derived as a special case of his general moment theory [Ukrain. Mat. Ž. **1** (1949), no. 2, 3–66; MR **14**, 56]. The author shows that the answer is negative for the circle group, and affirmative for the discrete group of integers. For Euclidean space of any dimension the problem is

open; however, by placing additional restrictions the author answers the question in the affirmative. His main result, stated for the $(x_1, x_2)$-plane, is as follows: Let $Q$ be the rectangle $|x_i| < a_i$, $i = 1, 2$; if (i) $f(x_1, 0)$ and $f(0, x_2)$ each have unique positive definite extensions along the $x_1$-axis and $x_2$-axis respectively, then there is a unique non-negative measure $dF$ such that

$$f(x_1, x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp i(x_1t_1 + x_2t_2)dF(t_1, t_2).$$

A similar result holds for the discrete Euclidean space. The theorem is the exact analogue of a theorem of the author on the two-parameter moment problem [Duke Math. J. **24** (1957), 481–498; MR **19**, 1047]. The author proves also that (i) may be replaced by (ii): there is an $\varepsilon > 0$ such that the restriction of $f(x_1, 0)$ to $(-2a_1 + \varepsilon, 2a_1 - \varepsilon)$ has a unique extension. The proofs are based on the following idea. Set $g(x) = \sum \zeta_k f(x - x_k)$, $h(x) = \sum \eta_j f(x - y_j)$, $(g, h) = \sum_1^n \sum_1^m \zeta_k \bar{\eta}_j f(y_j - x_k)$. In this way we get a Hilbert space $F$ of functions $g(x)$ defined on $Q$. Let $D_1$ be the linear manifold in $F$ such that $\partial g(x)/\partial x_1$ exists and belongs to $F$, and define the operator $A_1 g(x) = -i\partial g(x)/\partial x_1$ on $D_1$. The author proves that the operator $A_1^*$ exists and has self-adjoint extension $H_1$, and if $dE_1$ is the canonical spectral measure of $H_1$, $u = (1, 0)$ and $U(x_1 u) = \int_{-\infty}^{\infty} \exp(ix_1 t)dE_1(t)$, $-\infty < x_1 < \infty$, then $U(x_1 u)g(x) = g(x + x_1 u)$ for any $g \in F$, $x + x_1 u \in Q$. Similarly are defined $A_2$ and $U(x_2 v)$, $v = (0, 1)$. If

$$U(x_1 u)U(x_2 v) = U(x_2 v)U(x_1 u), \tag{1}$$

then $F(x) = (U(x_1 u)U(x_2 v)g_0, g_0)$, $g_0(x) = f(x - 0)$, provides the desired extension of $f$ to the whole plane. The main difficulty in the problem posed above is the proof of relation (1). The restrictions (i) or (ii) put on $f$ allow this relation to be proved by a careful examination of the domains of the operators. *M. Cotlar* (Buenos Aires)

876:

**Akutowicz, Edwin J. On extrapolating a positive definite function from a finite interval.** Math. Scand. **7** (1959), 157–169.

If $F(t)$ is continuous in a finite interval $-2a \leq t \leq 2a$, and positive-definite there, that is, $0 \leq t_j, t_k \leq 2a$ implies $\sum_{j,k} F(t_j - t_k)z_j\bar{z}_k \geq 0$; then, according to a generalization by M. Krein of the theorem of the reviewer, it is possible to continue $F(t)$ into a function $F_1(t)$ on the entire line $-\infty < t < \infty$ which is positive-definite everywhere. Such a continuation need not be unique, and the author contributes an interesting criterion as follows.

Consider the pre-Hilbert space $H_0$ of functions $\varphi(u) = \int_{-a}^{a} e^{iu\lambda}d\mu(\lambda)$ with inner product

$$(\varphi_1, \varphi_2) = \int_{-a}^{a}\int_{-a}^{a} F(\lambda_1 - \lambda_2)d\mu_1(\lambda_1)\overline{d\mu_2(\lambda_2)},$$

and denote by $H$ the resulting Hilbert closure after first reducing $H_0$ mod $(\varphi_1, \varphi_0) = 0$. Next if $T_0$ is a linear operator in $H_0$ given by $T_0\varphi = u\varphi(u)$ $(-\infty < u < \infty)$, then it first follows that $T_0$ has a minimal closed extension $T$ in $H$ which is Hermitian and whose deficiency index is $(0, 0)$ or $(1, 1)$.—Now, the theorem is that the extension of $F(t)$ to $F_1(t)$ is unique if and only if the operator $T$ is self-adjoint so that the deficiency index is $(0, 0)$. In the non-uniqueness case the classification of solutions is parallel to that of R. Nevanlinna for the power moment problem. *S. Bochner* (Princeton, N.J.)

877:

**Krein, M. G. Integral representation of a continuous Hermitian-indefinite function with a finite number of negative squares.** Dokl. Akad. Nauk SSSR **125** (1959), 31–34. (Russian)

Let $P_{\nu;a}$ denote the set of continuous $f$ for which $f(x) = \overline{f(-x)}$ on $(-a < x < a)$ and the diagonal form of $\sum_{j,k=1}^{n} f(x_j - x_k)\xi_k\bar{\xi}_j$ has $\leq \nu$ negative terms for all choices of $x_1, x_2, \cdots, x_n$ $(n = 1, 2, \cdots)$ in $[0, a)$ and has $\nu$ negative terms for some choice of the $x_i$. An integral representation for $f \in P_{\nu;a}$ is given. The basic steps are the following. (1) Let $Q(\lambda) = \lambda^\nu + q_1\lambda^{\nu-1} + \cdots + q_\nu$. For $\varphi \in C^{(\nu)}$, $\varphi(0) = \varphi'(0) = \cdots = \varphi^{(\nu-1)}(0) = 0$, $\varphi = 0$ in some $(a - \varepsilon, a]$, let $\psi = Q(-i(d/dx))\varphi$. Then let

$$P_{Q;a} = \{f \mid (\psi, \psi) = \int_0^a\int_0^a f(x - s)\psi(s)\overline{\psi(x)}dsdx \geq 0\}.$$

Via the theory of directrices for Hermitian operators an integral representation for $f \in P_{Q;a}$ is found. (2) For $f \in P_{\nu;\infty}$ there can be found a $Q$ such that $f \in P_{Q;\infty}$. (3) If $f \in P_{\nu;a}$, and if for sufficiently small $h > 0$,

$$\sum_{p,q=0}^{\nu-1} f((p - q)h)\xi_q\bar{\xi}_p$$

is negative, then $f$ has an extension to an $\tilde{f} \in P_{\nu;\infty}$.

*B. R. Gelbaum* (Princeton, N.J.)

## INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 761.

878:

**Miller, J. B. A symmetrical convergence theory for general transforms. II.** Proc. London Math. Soc. (3) **9** (1959), 451–464.

The present work is an investigation of certain technical questions arising in an earlier paper [same Proc. (3) **8** (1958), 224–241; MR **20** #3430]. These questions have to do with simplifying definition of the class $D_p^2$.

*J. Blackman* (Syracuse, N.Y.)

879:

**Zemanian, Armen H. On the pole and zero locations of rational Laplace transformations of non-negative functions.** Proc. Amer. Math. Soc. **10** (1959), 868–872.

Let $F(s)$ be a Laplace-Stieltjes transform which is a rational function. The author gives a sufficient condition, expressed in terms of the numbers of real and complex zeros and poles and their positions, for $F(s)$ to be completely monotonic.

*I. I. Hirschman, Jr.* (St. Louis, Mo.)

880:

**Radlow, James. On the double Laplace transforms of some Green's functions.** J. Math. Phys. **38** (1959/60), 203–205.

Let $g(x_1, x_2)$ be the Green's function of a linear second order elliptic equation:

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\right)g(x_1, x_2) = -\delta(x_1)\delta(x_2),$$

where $a_1$, $a_2$, and $a_3$ are arbitrary constants. The author

calculates the double Laplace transform (i.e., the transform over the first quadrant of the $x_1x_2$ plane). The result is explicit to within single integrals of known type. Two new Bessel function integrals, the transforms of the Helmholtz equation Green's functions, are given as examples. *I. I. Hirschman, Jr.* (St. Louis, Mo.)

881:

**Диткин, В. А.; Прудников, А. П. ★Операционное исчисление по двум переменным и его приложения. [Ditkin, V. A.; Prudnikov, A. P. Operational calculus in two variables and its applications.]** Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 178 pp. 4.60 rubles.

In this little treatise which is written for students, engineers and research workers an introduction to the theory of two-dimensional Laplace transformation is given. Much of its contents is taken from the book of Voelker and Doetsch [*Die zweidimensionale Laplace-Transformation*, Verlag Birkhäuser, Basel, 1950; MR **12**, 699] and the monograph of Poli and Delerue [*Le calcul symbolique à deux variables*, Gauthier-Villars, Paris, 1954; MR **16**, 243]. However, the basic theory is developed somewhat more fully. This booklet contains a valuable collection of formulae of about 80 pages. In these tables the Laplace-Carson version of the transformation formula is used in contrast to the tables given by Voelker and Doetsch.

*H. A. Lauwerier* (Amsterdam)

882:

**Ditkin, V. A. On the theory of operator calculus.** Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 15–17. (Russian)

The article constructs operational calculus for the operator $(d/dt)t(d/dt)$. For this purpose is defined a ring $C_2$ of twice differentiable functions on the half-line $0 \le t < \infty$, whose second derivatives are piecewise continuous. The product of two functions in this ring, $\varphi(t) \in C_2$ and $\psi(t) \in C_2$, is

$$\varphi(t) \cdot \psi(t) = \frac{d}{dt}\left\{ t \frac{d}{dt} \int_0^t d\xi \int_0^1 \varphi(x\xi)\psi[(1-x)(t-\xi)]dx \right\}.$$

Composition in the ring is defined in the usual way. It is proved possible to extend this ring to a quotient field. Following Mikusiński [*Rachunek operatorów*, Polskie Towarzystwo Mat., Warsaw, 1953; *Operational calculus*, Pergamon, New York, 1959; MR **16**, 243; **21** #4333], the author calls elements of the field operators. The operator $1/t$ is denoted $B$. It is shown that

$$B^{-1}\psi(t) = \int_0^t \xi^{-1} d\xi \int_0^t \psi(u)du.$$

If $\varphi(t) \in C_2$ and $\varphi(0) = 0$, then $B\varphi(t) = (d/dt)(td\varphi/dt)$.

Further, formulas of operational calculus for the operator $B$ are given, such as

$$\frac{B}{B^2+a^2} = \mathrm{ber}(2\sqrt{(at)}),$$

$$\frac{B}{(B-a)^{n+1}} = \frac{1}{n!}\left(\frac{1}{a}\right)^{n/2} I_n(2\sqrt{(at)}).$$

It is indicated that an operational calculus can be constructed starting from the integral transform

$$\bar{\varphi}(p) = 2\int_0^\infty \varphi(t) K_0(2\sqrt{(pt)})dt.$$

If the function $\varphi(t)$ is measurable and satisfies the condition $|\varphi(t)| < Q \exp(2\gamma_0 t^{1/2})$, where $Q$ and $\gamma_0$ are constants $> 0$, then the multiplication theorem holds: If $\varphi(t)$ corresponds to $\bar{\varphi}(p)$, $\psi(t)$ to $\bar{\psi}(p)$, then to the product $\bar{\varphi}(p)\bar{\psi}(p)$ corresponds the function

$$\omega(t) = \int_0^t d\xi \int_0^1 \varphi(x\xi)\psi[(1-x)(t-\xi)]dx.$$

It should also be mentioned that the author comments on the desirability of basing the Heaviside operational calculus on the product

$$a(t) * b(t) = \frac{d}{dt}\int_0^t a(t-\xi)b(\xi)d\xi.$$

This simplifies the Mikusiński theory, since constants need not be distinguished from constant functions.

*P. I. Kuznecov* (RŽMat **1958** #7879)

883:

**Arya, S. C. Convergence theorems and asymptotic properties of a generalized Stieltjes transform.** J. Indian Math. Soc. (N.S.) **22** (1958), 117–135.

This paper is concerned with the hypergeometric transform

$$\text{(i)} \qquad f(s) = As^{-1}\int_0^\infty F(t, s)d\alpha(t),$$

where

$$F(t, s) = F\left[\begin{matrix} 2m+1, 1 \\ m-k+3/2 \end{matrix}; -\frac{t}{s}\right], \quad A = \frac{\Gamma(2m+1)}{\Gamma(m-k+3/2)}.$$

When $2m+1 = \rho$ and $k-m = \frac{1}{2}$ the transform reduces to

$$f(s) = \Gamma(\rho)s^{\rho-1}\int_0^\infty (s+t)^{-\rho}d\alpha(t).$$

The first two sets of theorems are concerned with regions of convergence of the integral (i) and with properties of $\alpha(t)$ on the assumption that the integral does converge for one value of $s$. The remaining theorems express relations between the asymptotic forms of $f(s)$ and $\alpha(t)$. The results obtained in this paper are generalisations of sections of Chapter 8 of D. V. Widder's *The Laplace transform* [Princeton Univ. Press, 1941; MR **3**, 232].

*J. L. Griffith* (Kensington)

884:

**Griffith, James L. On some of the singularities of the Hankel transform.** J. Proc. Roy. Soc. New South Wales **93** (1959), 61–67.

The location and nature of some of the singularities of the function $F$, $F(z) = \int_0^\infty xJ_\nu(xz)f(x)dx$, are determined when, as $x \to \infty$, $f \sim Ax^\sigma e^{-bx}$, $\Re b \ge 0$. To do this the author replaces $J_\nu$ by its asymptotic representation as $x \to \infty$ and then uses Theorem 1, p. 256 in G. Doetsch, *Handbuch der Laplace-Transformation, Bd. 1*, Birkhäuser, Basel, 1950 [MR **13**, 230].

*N. D. Kazarinoff* (Moscow)

885:

**Colombo, Serge. ★Les transformations de Mellin et de Hankel: Applications à la physique mathématique.** Monographies du Centre d'Études Mathématiques en vue des Applications: B.—Méthodes de Calcul. Centre National de la Recherche Scientifique, Paris, 1959. 99 pp. 1000 francs.

This monograph treats the mutually connected integral transforms known as Fourier, Laplace, Mellin and Hankel transforms. The applications to ordinary differential equations, integral equations and partial differential equations of mathematical physics are considered in particular.

The first chapter gives a clear survey of the most essential characteristics of these transforms, concerning, e.g., the domains of convergence, the uniqueness of the corresponding inversion integral, and the applications to linear differential equations. Examples of Fourier cosine transforms, Fourier sine transforms, Fourier transforms proper, unilateral and bilateral Laplace transforms are represented in tables. Two additional tables of the elementary computation rules for the unilateral and bilateral Laplace transform demonstrate the simpler form of the latter.

The second chapter deals with the Mellin transform $M\{f(x)\}=\int_0^\infty x^{s-1}f(x)dx$, which reduces to the bilateral Laplace transform on substituting $x=\exp(-t)$. This reduction involves modified computation rules. A further table shows in particular the connection of Mellin transforms of Bessel functions with gamma functions. This transform is used for a study of all "Fourier kernels" $K(x)$ defined such that

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \int_0^\infty K(xz)dx\int_0^\infty K(xy)f(y)dy$$

holds under very general conditions. The relation $\kappa(s)\kappa(1-s)=1$ is proved in particular; $\kappa(s)$ here represents the Mellin transform of $K(x)$. This chapter closes with an interesting derivation of Poisson's summation formula.

The Hankel transform

$$\mathfrak{H}_\nu\{f(x)\} = \int_0^\infty x\mathfrak{J}_\nu(sx)f(x)dx$$

is discussed in the next chapter. The recurrence relation and the differential equation of the Bessel functions lead to special computation rules for this transform. The special transform for $\nu=n/2-1$ determines the $n$-dimensional Laplace integral of functions which are radially symmetric in an $n$-dimensional space. The Hankel transforms prove to be connected with the unilateral Laplace transform of $x^{\nu/2}f(x)$. Apart from a table of Hankel transforms, examples are given of functions which are self reciprocal with respect to this transform.

Chapter IV discusses the applicability of the Mellin and Hankel transform to functions of three variables with cylindrical symmetry. Examples concerning the potential and the diffusion equation are worked out. The final chapter V deals with integral equations of the first kind with a kernel of the type $\mathfrak{J}_\nu(r\zeta)$, and which have a different form for $0<r<r_0$ and $r>r_0$. The theory of the Mellin and Hankel transforms proves to be very useful here.

{We mention a serious printing error on page 51 where $\kappa(a)$ should read $\kappa(s)$ in (32).}

*H. Bremmer* (Eindhoven)

886:

**Cheng, Min-Teh; Chen, Yung-Hoo. Fractional integrals of functions of several variables with applications to the theory of approximation.** Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 639–641.

This is an announcement of results for which proofs will be given in the Science Reports of Peking University. Let $F(x)$ be defined and continuous on the Euclidean space of dimension $m$, periodic of period 1 in each variable, and let $\int_D F(x)dx=0$, where $D$ is the unit cube. Let $n=(n_1,\cdots,n_m)$, $x=(x_1,\cdots,x_m)$, $nx=n_1x_1+\cdots+n_mx_m$, $|n|^2=n_1^2+\cdots+n_m^2$, $F(x)\sim\sum c_ne^{inx}$ and $A_\nu(x)=\sum_{|n|^2=\nu}c_ne^{inx}$. Define the fractional integral of $F(x)$ of order $\alpha$ to be

$$I_\alpha^*[F(x)] = \lim_{R\to\infty}\sum_{\nu\leqq R^2}'\left(1-\frac{\nu}{R^2}\right)^{\sigma_m}\frac{A_\nu(x)}{(2\pi)^\alpha\nu^{\alpha/2}e^{i\pi\alpha/2}},$$

where $\sigma_m=[m-1/2]+1$ and $\alpha\geqq 0$. For $0<\alpha<1$ this coincides with the definition of the fractional integral given by M. Riesz [Acta Math. 81 (1949), 1–223; MR 10, 713]. Define the mean modulus of continuity of $F(x)$ as

$$\omega^*(\rho) = \max_{\substack{x\in D\\ 0\leq t\leq\rho}}\left|\frac{1}{S(t)}\int_{S(x;t)}\{F(y)-F(x)\}dy\right|,$$

where $S(t)$ is the surface area of the sphere $S(x;t)$: $|y-x|^2=t^2$. If $\omega^*(\rho)=O(\rho^\alpha)$, $F$ is said to be Lip$^*\alpha$. Theorem: If $F\in$ Lip $\alpha$ $(0<\alpha\leqq 1)$, then for $0<\beta<1$ $I_\beta^*[F]\in$ Lip$^*(\alpha+\beta)$. If $0<\alpha+\beta<1$, then $I_\beta^*[F]\in$ Lip $(\alpha+\beta)$.

Define the fractional Laplace operator $\Delta^\beta$ of order $\beta$ as $\Delta^\beta F(x)=\Delta I_{2(1-\beta)}^*[F(x)]$. Theorem: If $F\in$ Lip$^*\alpha$, then for $1<2\beta<\alpha\leqq 2$, $\Delta^\beta F\in$ Lip$^*(\alpha-2\beta)$. Theorem: If $\Delta^\beta F(x)=G(x)$ $(0<\beta<1)$ is continuous as $x$ ranges over two-dimensional Euclidean space, then $I_{2\beta}^*[G(x)]=F(x)$.

The remainder of the paper is concerned with the degree of approximation of a periodic function $f$ of two variables with $2p$ continuous partial derivatives by means of spherical $(C,\delta)$ sums of its Fourier series in terms of the modulus of continuity of $\Delta^p f$, where $\Delta$ is the ordinary Laplace operator. *A. Devinatz* (St. Louis, Mo.)

## INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 905.

887:

**Sadowska, D. Sur une équation intégro-différentielle de la théorie de la conductibilité.** Ann. Polon. Math. 7 (1959), 81–92.

The temperature distribution $T$ in the steady state within a conducting and radiating medium is given by an integro-differential equation

$$\Delta T = f(A,T)+\iiint_D F(A,B,T(B))/r_{AB}^2dv_B.$$

The bounded domain $D$ in $E_3$ is assumed to have Liapounoff boundary $S=\bar{D}-D$ and a solution $T(A)$ is sought in $D$ which is continuous in $\bar{D}$ and assumes the value zero on $S$.

The first part of the paper is devoted to showing that under certain conditions on $f$ and $F$ and relations between these and the diameter $L$ of the domain $D$, a unique solution can be shown to exist by the method of successive substitutions. In the second part a solution is shown to exist under weaker hypotheses by using the Schauder fixed point theorem. The assumptions on which the latter is based are: $f(A,u)$ is uniformly Hölder continuous in $A\in\bar{D}$, $|u|\leq R$, while $F(A,B,u)$ is continuous in $A\in\bar{D}$, $B\in\bar{D}$, $|u|\leq R$ and uniformly Hölder continuous in $A$ and

$u$. Finally, the diameter $L$ of $D$ is assumed to satisfy a relation of the form $CML^2(L|\log L|+L+1)\leq R$ where $M=\sup(|f|, |F|)$ and $C$ is an absolute constant.

*A. N. Milgram* (Minneapolis, Minn.)

888:

**Doležal, Václav. On the use of distributions in the theory of linear dynamical systems.** Apl. Mat. **4** (1959), 405–440. (Czech. Russian and English summaries)

The first part of this paper gives an elementary exposition of the fundamentals of the theory of distributions, noting in detail those distributions for which a Laplace transform can be defined. The author then employs this theory to extend his previous study [Apl. Mat. **4** (1959), 1–17; MR **21** #281] of systems of ordinary linear integro-differential equations with real constant coefficients of the type occurring in the analysis of lumped parameter electrical circuits,

$$\sum_{k=1}^{r}\{\alpha_{ik}x_k(t)+\beta_{ik}\dot{x}_k(t)+\gamma_{ik}\int_0^t x_k(\tau)d\tau\}+q_i = f_i(t) \qquad (i = 1, 2, \cdots, r),$$

the right members now being taken to be distributions.

*J. F. Heyda* (Cincinnati, Ohio)

889:

**Melzak, Z. A. The positivity sets of the solutions of a transport equation.** Michigan Math. J. **6** (1959), 331–334.

Let $Df(x,t)=0$, $f(x,0)=g(x)\geq 0$, $(x,t)\geq 0$, be an initial-value problem whose operator $D$ and initial function $g(x)$ are such that (i) there exists a unique continuous solution $f(x,t)$ valid for $(x,t)\geq 0$, (ii) $f(x,t)$ is analytic in $t$ for each $x$, (iii) $f(x,t)\geq 0$ $[>0]$ if $g(x)\geq 0$ $[>0]$. Let $P=\{x|g(x)>0\}$, $Z=\{x|g(x)=0\}$, and $Z_n=\{x|\partial^k f(x,0)/\partial t^k=0\ (k=0, 1, \cdots, n-1),\ \partial^n f(x,0)/\partial t^n>0\}$ $(n=1, 2, \cdots)$; $Z_n$ is called the $n$th positivity set. In this note, an example of a nonlinear integrodifferential operator $D$ is given for which the sets $Z_n$ can be completely described in terms of $Z$ and $P$ alone, this operator being a special case of a more general class of operators considered previously by the author [Trans. Amer. Math. Soc. **85** (1957), 547–560; MR **19**, 428].

*R. S. Varga* (Cleveland, Ohio)

## FUNCTIONAL ANALYSIS

See also 768, 811, B1068.

890:

**Pták, Vlastimil. A combinatorial theorem on systems of inequalities and its application to analysis.** Czechoslovak Math. J. **9**(84) (1959), 629–630. (Russian summary)

The theorem proved here yields theorems connecting convergence and uniform convergence in double sequences and may be applied to obtain results on weak compactness in Banach spaces. Given a set $S$, denote by $C$ the set of all functions $\lambda$ with domain $S$ such that (1) $\lambda(s)\geq 0$ for every $s\in S$, (2) the set $N(\lambda)$ of those $s\in S$ where $\lambda(s)>0$ is finite, and (3) $\sum_{s\in S}\lambda(s)=1$. If $\varepsilon>0$, $H\subset S$, and $W$ is a system of finite subsets of $S$, denote by $C(\varepsilon, H, W)$ the set of all $\lambda\in C$ such that $N(\lambda)\subset H$ and $\sum_{s\in w}\lambda(s)<\varepsilon$ for each $w\in W$. The theorem proved is the following. Let $S$ be an infinite set and let $T$ be a system of finite subsets of $S$. The following two conditions are equivalent to each other. (i) There exists an infinite $H\subset S$ and $\varepsilon>0$ such that $C(\varepsilon, H, T)=0$; (ii) there exists a strictly increasing sequence of finite sets $B_n\subset S$ and a sequence $t_n\in T$ such that $B_n\subset t_n$.

*E. J. Cogan* (Bronxville, N.Y.)

891:

**Pták, Vlastimil. Biorthogonal systems and reflexivity of Banach spaces.** Czechoslovak Math. J. **9** (84) (1959), 319–326. (Russian summary)

The following theorems are proved en bloc. (Note: A biorthogonal system $S=\{e_i; f_j\}$, where $(e_i, f_j)=\delta_{ij}$, is called bounded if there is a constant $\mu$ such that $\|e_i\|$, $\|f_j\|\leq\mu$, all $i, j$.) 1. A Banach space $E$ is reflexive if and only if, for every bounded biorthogonal system $\{e_i; f_j\}$, the sums $\sum_{i=1}^n e_i$ are unbounded. 2. $E$ is reflexive if and only if for every bounded biorthogonal system $\{e_i; f_j\}$ the sums $\sum_{j=1}^n f_j$ are unbounded. 3. The following statements are equivalent. (a) $E$ is non-reflexive. (b) There is a bounded biorthogonal system $S_1=\{e_i; f_j\}$ and an $\omega>0$ such that if $\alpha_n\uparrow 0$ or if $\alpha_n\downarrow 0$ then $x=\sum_{i=1}^\infty \alpha_i e_i$ exists and $\|x\|\leq\omega|\alpha_1|$. (c) Let $B(S)=$ weak* closure of $f_j$ in $E^*$, $B(S)^0=(B(S))_\perp=\{x|x\in E, (x, B(S))=0\}$. There exists a bounded $S_2$ such that, considered as a biorthogonal system in $E/B(S_2)^0$, $S_2$ enjoys the property: there is an $\omega>0$ such that if $0\leq\lambda_i$, $\sum_{i=1}^\infty\lambda_i<\infty$, then $\sum_{i=1}^\infty\lambda_i e_i\geq\omega\sum_{i=1}^\infty\lambda_i$.

The key to the theorems lies in constructing for a given non-reflexive $E$ a system $S$ for which 3(b) and 3(c) are true. Once this is achieved, the remaining syllogisms follow readily. The root idea is then the following. Choose $r\in E^{**}|E$. Then for $\delta>0$, by induction construct sequences $\{b_i\}\subset E$, $\{y_j\}\subset E^*$ such that $\|b_i\|\leq 1+\delta$, $\|y_j\|=1$, $(b_i, y_j)=\beta_j>\frac{1}{2}\delta_j>0, j\leq i$, $(b_i, y_j)=0, j>i$, where $\delta_j=\sup\{(r, y)|\ \|y\|\leq 1, y\in E_{j-1}=$ linear space spanned by $b_1, b_2, \cdots, b_{j-1}\}$. Then there is a $\beta>0$ such that $\delta_j\geq 2\beta$. Set $e_i=b_i-b_{i-1}$, $f_j=(1/\beta_j)y_j$. $\{e_i; f_j\}$ is a system $S_1$ for 3(b). Set $g_i=b_i$, $h_j=(1/\beta_j)y_j-(1/\beta_{j+1})y_{j+1}$. $\{g_i; h_j\}$ is a system $S_2$ for 3(c). $S_1$ is also a system for which the sums $\sum_{i=1}^n e_i$ are bounded. The remaining arguments are less involved. Finally in $L_E[0,1]$ the author constructs an $S$ for which there are two positive numbers $\alpha\geq\beta>0$ such that for $\lambda_i\geq 0$, $\alpha\sum_{i=1}^n\lambda_i\geq\sum_{i=1}^n\lambda_i e_i\geq\beta\sum_{i=1}^n\lambda_i$, all $n$.

*B. R. Gelbaum* (Princeton, N.J.)

892:

**Seidman, Thomas I. Linear transformations of a functional integral. I.** Comm. Pure Appl. Math. **12** (1959), 611–622.

Let $A_n$ be a linear transformation on $R_n$ and let $f(x_n)$ be summable over $R_n$ with respect to

$$d\mu_n=(2\pi)^{-n/2}\sigma^{-n}\exp\{-(2\sigma^2)^{-1}(\xi_1^2+\cdots+\xi_n^2)\}d\xi_1\cdots d\xi_n,$$

where $x_n=(\xi_1, \cdots, \xi_n)\in R_n$ and $\sigma$ is a fixed positive number. Then the function

$$g(x_n) = f(x_n+A_nx_n)\Delta_n E(x_n),$$

where $\Delta_n=\det(I_n+A_n)$ and

$$E(x_n) = \exp\{-(2\sigma^2)^{-1}(x, \{A_n+A_n^*+A_n^*A_n\}x)\},$$

is also summable over $R_n$ with respect to the same measure, and $I_n(g)=I_n(f)$, where $I_n$ denotes the integral with respect to this measure.

This theorem is extended to infinite dimension. The space considered is the space of all sequences, obviously

an extension of the Hilbert space $l_2$, and the countably additive measure $\mu$ in this space is defined such that for the cylindrical sets, depending only on the first $n$ variables, $\mu$ coincides with $\mu_n$ above. The theorem holds now for this measure $\mu$, and for linear transformations $A$ satisfying

$$2\{\tau(A^*A)\}^{1/2}+|\tau(A)| < \infty$$

($\tau(A)$ is the trace of $A$), and $\Delta(A)>0$, where $\Delta(A)$ is the appropriately defined determinant of $I+A$.

*A. C. Zaanen* (Pasadena, Calif.)

893:

**Mrówka, S. A remark on linear functionals.** Colloq. Math. **7** (1959), 53–55.

Let $C(X)$ be the set of all continuous real-valued functions on a completely regular space $X$ considered as a linear space with the $p$-topology (pointwise convergence). Hewitt [Fund. Math. **37** (1950), 161–189; MR **13**, 147] has shown that every continuous linear functional on $C(X)$ is a finite linear combination of evaluations at points of $X$. This he derived as one of many consequences of the theory of integral representations of linear functionals on $C(X)$ (in various topologies) which he developed there. In the present paper the author gives a proof of Hewitt's theorem which is direct and involves no integration theory.

*B. Yood* (Eugene, Ore.)

894:

**Il'in, V. P. Some functional inequalities of the type of theorems of imbedding with a weight.** Dokl. Akad. Nauk SSSR **129** (1959), 983–985. (Russian)

Let $D$ be a region in Euclidean $n$ space which is such that for each point in $D$ there is a spherical sector of fixed shape with vertex at this point, and let $f$ be sufficiently often continuously differentiable in $D$. Then inequalities are given relating $L^p$ norms for derivatives of $f$ multiplied by powers of distances from fixed points in $D$ to similar norms referred to a section of $D$ by an $m$-dimensional hyperplane.

*J. L. B. Cooper* (Cardiff)

895:

**Gel'man, I. V. A test for complete continuity of an imbedding operator.** Dokl. Akad. Nauk SSSR **129** (1959), 242–245. (Russian)

Let $\Omega$ denote a domain in Euclidean $n$-space, $\omega$ an $s$-dimensional manifold lying in $\bar{\Omega}$. Geometrical simplicities required to obtain the standard Sobolev inclusion theorems are assumed. (It suffices, for instance, that $\Omega$ be bounded and convex and that $\omega$ possess smooth parametrization having bounded derivatives. Considerably weaker conditions will do. For an excellent exposition of the basic Sobolev theorems in the usual $L^p$ context see Kantorovič, Uspehi Mat. Nauk **11** (1956), no. 2 (68), 3–29 [MR **20** #5432].) The author continues the program of pushing through similar inclusion theorems in the context of Orlicz spaces, the results of this paper consisting, in part, of extensions and improvements of the author, same Dokl. **122** (1958), 547–550 [MR **21** #301a]. Both results and proofs are too complicated to summarize coherently in a brief review; the general flavor of the investigations is indicated in the following statement of the criterion to which the title alludes.

Let $p>0$ and let $M(u)$ be an $N$-function satisfying $M(u)\sim|u|^p\,|\log u|^{2(p-1)+\varepsilon}$ for some $\varepsilon>0$. (Terminology and notation relating to Orlicz spaces are those of Krasnosel'skiĭ and Rutickiĭ [*Vypuklye funkcii i prostranstva Orliča*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958; MR **21** #5144].) For positive integral $k$ let $F_M{}^{(k)}$ denote the space of functions on $\Omega$ all of whose $k$th order generalized derivatives (in the sense of Sobolev) exist and belong to the space $L_M{}^*$. Suppose finally that $n-kp<s$. Then for any $\sigma>0$ we have, set-wise, $F_M{}^{(k)}\subset L_{M_1}{}^*(\omega)$ where $M_1(u)\sim|u|^q\,|\log u|^{-1-\sigma}$ and $q=sp/n-kp$, and the inclusion operator is completely continuous.

*A. Brown* (Houston, Tex.)

896:

**Berezanskiĭ, Yu. M. On the boundary value problems for general partial differential operators.** Dokl. Akad. Nauk SSSR **122** (1958), 959–962. (Russian)

The reviewer's quoted results [e.g., Rend. Accad. Sci. Fis. Mat. Napoli (4) **2** (1932), 29–40; C. R. Acad. Sci. Paris **204** (1937), 1022–1024; Bul. Inst. Politehn. Iași. **3** (1948), 153–155; MR **10**, 719] concerning various boundary value problems $[A(X)u'+\lambda B(X)u]'+\lambda[B(X)u'+C(X)u]=0$, $u|_R=0$ (here $R$ is a $2n$-dimensional rectangular domain and the symbol $'$ designates total differentiation in the sense that $u'=\partial^{2n}u/\partial x_1\partial x_2\cdots\partial x_{2n}$), have been improved by A. Rosenblatt [C. R. Acad. Sci. Paris **197** (1933), 1278–1280], M. Salvadori [Ann. Scuola Norm. Sup. Pisa (2) **5** (1936), 51–72], and F. Manaresi [Rend. Sem. Mat. Univ. Padova **23** (1954), 163–213; MR **16**, 138]. In the last years various research workers have intensively explored different classes of boundary value problems related to non-elliptic differential equations [see the bibliography in the reviewed paper]. In this paper the author continues the work toward the resolution of the boundary value problems for general partial differential equations.

Let $G$ be a domain of an $n$-dimensional space with fragmentary-regular boundary, $W_2{}^l$ a Sobolev space of functions, each element of $W_2{}^l$ being $L_2$-integrable with its derivatives of order $l$, and $(u, v)_l$ the scalar product on $W_2{}^l$. Consider first the determination of the operator $I$ defined by $(f, u)_0=(If, u)_l$ for every $f\in W_2{}^0\,(=L_2)$. Assuming $f, g\in L_2$, $(f, g)_{-l}=(If, Ig)_l$, where $\|f\|_{-l}\leq\|f\|_0$, it is possible to define the Hilbert space $W_2{}^{-l}$. Relating the second-order partial differential operator

$$L[u]=\sum_{j,k=1}^{n}a_{jk}(x)\frac{\partial^2u}{\partial x_j\partial x_k}+\sum_{j=1}^{n}a_j(x)\frac{\partial u}{\partial x_j}+a(x)u \qquad (x\in G),$$

with the scalar product $(u, v)_L=(L[u], L[v])_{-2}$, one defines the corresponding Hilbert space $H_L$. The continuity of the operator $L: H_L\to W_2{}^{-2}$ is proved. Defining a corresponding generalized solution of the equation $Lu=f$, the following theorem is proved: In order that there exist a generalized solution for a boundary value problem for the equation $Lu=f$ for every $f\in W_2{}^0$, it is necessary and sufficient that every classical (i.e., belonging to $W_2{}^2$) solution $v$ of the homogeneous formally adjoint equation $L^*[v]=0$ satisfying adjoint boundary conditions vanishes. Several (8) applications are discussed at the end of the paper.

*D. Mangeron* (Iași)

897:

**Hirata, Yukio; Ogata, Hayao. On the exchange formula for distributions.** J. Sci. Hiroshima Univ. Ser. A **22**, 147–152 (1958).

Let $S$ and $T$ be temperate distributions on $R^n$ with Fourier transforms $\hat{S}$ and $\hat{T}$. The "exchange formula" in question reads $(S*T)^{\wedge} = \hat{S}\cdot\hat{T}$. Conditions under which this formula is valid are given by Schwartz [*Théorie des distributions*, Tome II, Act. Sci. Ind., no. 1122, Hermann, Paris, 1951; MR **12**, 833], whilst the reviewer noted slight extensions in connection with problems about spans of translates [Proc. Amer. Math. Soc. **4** (1953), 888–894; MR **16**, 354]. By adapting a definition of the convolution ascribed to Chevalley, the authors investigate the exchange formula systematically. The modified concept of convolution amounts to saying that the $\mathscr{S}'$-convolution $S \circledast T$ exists if and only if the function $(S*\varphi)\cdot(\check{T}*\psi)$ is integrable for any two indefinitely differentiable and rapidly decreasing functions $\varphi$ and $\psi$; when this is the case there exists a unique distribution $U$, necessarily temperate, such that

$$\int (S*\varphi)\cdot(\check{T}*\psi)dx = \langle U*\varphi, \psi\rangle$$

identically in $\varphi$ and $\psi$, and $S \circledast T$ is defined to be this $U$. Consistency with other definitions and associativity of $\circledast$ is discussed. The main theorem states that the exchange formula holds whenever the $\mathscr{S}'$-convolution exists.

*R. E. Edwards* (Reading)

898:

**Bessaga, C.; Pełczyński, A.; Rolewicz, S. Some properties of the space ($s$).** Colloq. Math. **7** (1959), 45–51.

Let $(s)$ denote the locally convex vector space of all real sequences $x=\{a_n\}$ with the topology determined by the semi-norms $\|x\|_n = \max_{i\leq n}|a_i|$ $(n=1, 2, 3, \cdots)$. This paper deals with characterizations of $(s)$. The following are typical theorems. (I) A complete metrizable locally convex vector space $X$ is isomorphic to $(s)$ if and only if every infinite-dimensional subspace of $X$ contains a subspace isomorphic to $(s)$. (II) Let $X$ be a space of type $(F)$ (in the terminology of Banach) with the norm $\|x\|$ (in general not homogeneous). $X$ contains a subspace isomorphic to $(s)$ if and only if, for every $\varepsilon>0$, there is an element $x\neq 0$ in $X$ such that $\|tx\|<\varepsilon$ for all real $t$. (III) Let $X$ be a space of type $(F)$ with a basis $\{e_n\}$. If every infinite-dimensional subspace of $X$ contains a subspace isomorphic to $(s)$, then $X$ is isomorphic to $(s)$. The paper is closely related to an earlier one by the first two authors [Bull. Acad. Polon. Sci. Cl. III **5** (1957), 375–377, XXX; MR **19**, 562].

*Ky Fan* (Detroit, Mich.)

899:

**Mamedov, R. G. The asymptotic value of the approximation of differentiable functions by linear positive operators.** Dokl. Akad. Nauk SSSR **128** (1959), 471–474. (Russian)

Soit $H^{(2)}(x)$ l'espace vectoriel ordonné des fonctions réelles définies sur la droite réelle, bornées et deux fois dérivables au point $x$. Soit $(L_n)$ une suite d'opérateurs linéaires positifs sur $H^{(2)}(x)$. Si $f\in H^{(2)}(x)$, la valeur de la fonction $L_n(f)\in H^{(2)}(x)$ au point $x$ est notée $L_n[f(t);x]$. L'A. affirme que sous certaines conditions, si pour les fonctions $f_k(t)=t^k$ $(k=0, 1, 2)$ on a $L_n[f_k(t);x]\to f_k(x)$, alors cette relation est vraie pour toute fonction $f\in H^{(2)}(x)$. Par exemple, théorème 2: Si

$$L_n[1;x] = 1+o[1/\varphi_2(n)],$$
$$L_n[t^k;x] = x^k+\psi_k(x)/\varphi_k(n)+o[1/\varphi_2(n)],$$

où $\varphi_k(n)\to\infty$ $(k=1, 2)$, si $\varphi_2(n)/\varphi_1(n)\to 0$ et si pour un $m>1$ on a $\lim_{n\to\infty}\varphi_2(n)L_n[(t-x)^{2m};x]=0$, alors

$$L_n[f(t);x]-f(x) = f''(x)\psi_2(x)/(2\varphi_2(n))+o[1/\varphi_2(n)]$$

quel que soit $f\in H^{(2)}(x)$. On applique ensuite ce théorème à certains opérateurs particuliers.

*N. Dinculeanu* (Bucharest)

900:

**Mamedov, R. G. On the order of the approximation of functions by linear positive operators.** Dokl. Akad. Nauk SSSR **128** (1959), 674–676. (Russian)

Soit $E$ un espace de Banach ordonné formé de fonctions réelles définies sur une partie $A$ de l'espace euclidien $R^k$. Soit $W$ un opérateur positif sur $E$. Si $f\in E$, la valeur de la fonction $W(f)\in E$ au point $x\in A$ est notée $W[f;x]$ ou $W[f(t);x]$. L'A. énonce quelques théorèmes de la forme suivante: sous certaines conditions on a $\|W(f)-f\| \leq 2\omega(\lambda)$, où $\lambda$ dépend de $W$ et $\omega$ dépend de $f$. Par exemple: (1) Si $E=C[a, b]$ et si $W[1;x]=1$, on prend $\lambda = \sup\{W[|t-x|;x], x\in[a, b]\}$ et $\omega(\lambda)=\sup\{|f(x+h)-f(x)|, |h|\leq\lambda\}$. (2) Si $E=L_p(-u, u)$, $0<u\leq\infty$, $1\leq p\leq\infty$ et si $W[f;x]=\int_{-u}^{u} f(t)K(t-x)dt$ où $K\in L_q(-u, u)$, $1/p+1/q=1$, on prend $\lambda=\int_{-u}^{u}|t|K(t)dt$ et

$$\omega(\lambda) = \sup\left\{\left(\int_{-u}^{u}|f(x+h)-f(x)|^p dx\right)^{1/p}, |h|\leq\lambda\right\}.$$

On applique ensuite ces théorèmes à certains opérateurs particuliers.

{Remarque du rapporteur: L'A. énonce ses théorèmes pour une suite $(W_n)$ d'opérateurs, ce qui n'est pas nécessaire.}

*N. Dinculeanu* (Bucharest)

901:

**Ivanov, V. V. Approximate solution of Riemann's boundary value problem for systems of $n$-functions.** Dokl. Akad. Nauk SSSR **129** (1959), 27–29. (Russian)

Let $A$ be a completely continuous operator in a Hilbert space, and let $\lambda_0$ be an eigenvalue of $A^{-1}$ so that $(E-\lambda A)^{-1} = \sum_1^k A_j(\lambda-\lambda_0)^{-j}+B(\lambda)$ with $B(\lambda)$ regular at $\lambda_0$. Let $P_m$ be a sequence of projections converging to $E$ for $m\to\infty$. It is observed that $2\pi i A_j$ can be obtained as the limit of $(E-\lambda_0 A)^{j-1}\int_{|\lambda-\lambda_0|=\varepsilon}(E-\lambda P_m A)^{-1}d\lambda$ as $m\to\infty$, provided $\varepsilon>0$ is small enough. Applications to the numerical solution of Riemann's problem when it has non-trivial null solutions.

*L. Gårding* (Lund)

902:

**Gilbert, Richard C. The denseness of the extreme points of the generalized resolvents of a symmetric operator.** Duke Math. J. **26** (1959), 683–691.

A generalized resolvent of a closed symmetric operator $A$ in a Hilbert space $\mathfrak{H}$ is an operator $R(\lambda)=P\tilde{R}(\lambda)$ defined for $I(\lambda)\neq 0$, where $\tilde{R}(\lambda)$ is the resolvent of a self-adjoint extension of $A$ in a space $\tilde{\mathfrak{H}}\supset\mathfrak{H}$, and $P$ is the projector from $\tilde{H}$ to $H$. It is proved that any $R(\lambda)$ can be approximated in norm uniformly over any compact set of nonreal $\lambda$ by a sequence of generalized resolvents $\{R_n(\lambda)\}$ each of which corresponds to an extension of $\mathfrak{H}$ to a space which is finite dimensional mod $\mathfrak{H}$.

*J. L. B. Cooper* (Cardiff)

903:

**Sz.-Nagy, Béla; Foiaş, Ciprian. Une relation parmi les vecteurs propres d'un opérateur de l'espace de Hilbert et de l'opérateur adjoint.** Acta Sci. Math. Szeged **20** (1959), 91–96.

The known fact that for a normal bounded operator $T$ in Hilbert space $Tx=\lambda x$ if and only if $T^*x=\bar{\lambda}x$, is extended as follows. If $T$ is a non-normal bounded operator and $S$ is a spectral set in the sense of von Neumann [Math. Nachr. **4** (1951), 258–281; MR **13**, 254] whose boundary $\partial S$ is a closed Jordan curve, then $\{x|Tx=\lambda x\}=\{x|T^*x=\bar{\lambda}x\}$ for all $\lambda\in\partial S$ (of course this reduces to $\{0\}=\{0\}$ unless $\lambda$ is an eigenvalue of $T$, so the theorem asserts that the conjugate of some of the point spectrum of $T$ is point spectrum for $T^*$). The proof makes use of the normal dilation of $T$, introduced by the authors in same Acta **19** (1958), 26–45 [MR **21** #2188]. By an elementary argument they prove also that if $S$ is a spectral set for $T$, $\lambda\in\partial S$, and there exist circles in the complement of $S$ whose centers $\lambda_n\to\lambda$ and with radii $\sim|\lambda_n-\lambda|$, then $\{x|Tx=\lambda x\}=\{x|T^*x=\bar{\lambda}x\}$. Finally they give a more general analytic condition on points $\lambda\in S$ which yields this equality and which is satisfied in both of the above cases.

*M. Schreiber* (Ithaca, N.Y.)

904:

**Evgrafov, M. A. Theorems analogous to the Phragmén-Lindelöf theorem.** Dokl. Akad. Nauk SSSR **126** (1959), 478–481. (Russian)

Verf. gibt ohne Beweis vier Sätze folgender Art: $H$ sei ein Banach-Raum und $A(x,\lambda)=\sum_{k=0}^{n}A_k(x)\lambda^k$ ein Operator in $H$, $0\leq x<\infty$, $\lambda$ komplex. $A(x,\lambda)$ habe, neben anderen Eigenschaften, in $\alpha<\mathrm{Re}\,\lambda<\beta$ bei festem $x$ nur für $m$ Zahlen $\lambda_1(x),\cdots,\lambda_m(x)$ Null-Lösungen mit endlichdimensionaler Lösungsmannigfaltigkeit, und es gelte $\lim_{x\to\infty}\lambda_\nu(x)=\lambda_\nu$. Jede Lösung von $A(x,d/dx)u(x)=0$, $u\in H$ mit $\|u(x)\|<Me^{(\beta-\delta)x}$ wobei $\delta>0$, hat die Form $u(x)=v(x)+\sum_{\nu=1}^{m}\sum_{p=1}^{p_\nu}C_{\nu p}u_{\nu p}(x)$, wobei $\|v(x)\|<M_\varepsilon e^{(\alpha+\varepsilon)x}$ für jedes $\varepsilon>0$, und $u_{\nu p}(x)$ eine Lösung ist mit

$$\limsup x^{-1}\ln\|u_{\nu p}(x)\|=\mathrm{Re}\,\lambda_\nu.$$

Für Beweise verweist Verf. auf Ideen einer früheren Arbeit [Trudy Moskov. Mat. Obšč. **5** (1956), 89–201; MR **18**, 472].

*F. Huckemann* (Giessen)

905:

**Seeley, R. T. Singular integrals on compact manifolds.** Amer. J. Math. **81** (1959), 658–690.

This paper treats two problems concerning singular integral operators on compact manifolds. The first problem is to develop an "approximate functional calculus" for singular integral operators. To each singular integral operator $T$ is associated a function $\sigma(T)$ on the cotangent bundle; $\sigma(T)$ is called the symbol of $T$. The author proves that $\sigma$ is a homomorphism onto and that $\sigma(T)=0$ if and only if $T$ is completely continuous.

The second problem is the representation of differential operators by means of a singular integral and an operator related to the Laplace-Beltrami operator $\Delta$. The author constructs an operator $\Lambda$ which has the property that $\Lambda^2=L-\Delta$ where $L$ is a real number; he then shows that if $V$ is a differential operator of order $m$ then there exists a singular integral operator $H$ such that $V=H\Lambda^m$.

While carrying out the investigation of these problems the author gives elegant proofs of some of the basic properties of singular integrals and of the Laplace-Beltrami operator.

*J. J. Kohn* (Waltham, Mass.)

906:

**Želobenko, D. P. Linear representations of the Lorentz group.** Dokl. Akad. Nauk SSSR **126** (1959), 935–938. (Russian)

Let $\gamma$ be a continuous representation of the Lorentz group $G$ by bounded operators in a Banach space $B$. If $B$ contains a finite sequence $0\subseteq B_0\subseteq B_1\subseteq B_2\subseteq\cdots\subseteq B_n$ of closed invariant subspaces such that the representations $\gamma_j$ defined in the difference spaces $B_j/B_{j-1}$ are all irreducible then $\gamma$ is said to be of finite rank and to have the form $\gamma_1\to\gamma_2\to\gamma_3\to\cdots\to\gamma_n$. When $\gamma$ is not unitary one cannot conclude at once that it is a direct sum of the $\gamma_j$, and there are examples showing that it need not be such a direct sum. Thus knowledge of all irreducible representations of $G$ does not immediately imply knowledge of all representations of $G$ of finite rank. In this note theorems are announced giving a rather complete analysis of the problem of finding all representations of the form $\gamma_1\to\gamma_2\to\cdots\to\gamma_n$ for each given $n$-tuple $\gamma_1,\gamma_2,\cdots,\gamma_n$. According to theorem 1 every representation of finite rank can be decomposed as a direct sum unless either (a) all $\gamma_j$ are equivalent, or (b) $n=2$, $\gamma_2$ is finite dimensional, and $\gamma_1$ is a certain infinite dimensional representation depending upon $\gamma_2$. Moreover in each indecomposable case there is an algebra whose finite dimensional representations are in one-to-one correspondence with the possible representations of $G$ having the given components.

*G. W. Mackey* (Cambridge, Mass.)

907:

**Nelson, Edward; Stinespring, W. Forrest. Representation of elliptic operators in an enveloping algebra.** Amer. J. Math. **81** (1959), 547–560.

Let $G$ be a Lie group, $\mathfrak{E}$ the universal enveloping algebra of its right-invariant Lie algebra. Let $L\to L^+$ be the conjugate linear anti-automorphism of $\mathfrak{E}$ such that $X^+=-X$ for $X$ in the Lie algebra. Let $T$ be a strongly continuous representation of $G$ on a Banach space $\mathfrak{B}$. For $f\in L_1(G)$ let $T(f)=\int f(\sigma)T(\sigma)d\sigma$. Let $dT(L)$ be the operator defined on the Gårding subspace of $\mathfrak{B}$ (i.e., the linear manifold spanned by $T(\varphi)x$ with $x\in\mathfrak{B}$ and $\varphi\in C_0^\infty(G)$, the class of all infinitely differentiable functions on $G$ with compact support) by $dT(L)(T(\varphi)x)=T(L\varphi)x$. Let $T=U$ be unitary and $L$ be elliptic as a partial differential operator on $G$. Then the closure of $dU(L^+)$ is the adjoint of $dU(L)$; if $L^+=L$, and $M\in\mathfrak{E}$ is such that $dU(M^+M)$ commutes with $dU(L)$, then the closure of $dU(M^+)$ is the adjoint of $dU(M)$. $U(f)$ is completely continuous for each $f\in L_1(G)$ if and only if the closure of $d(\Delta-1)$ has a completely continuous inverse, where $\Delta=X_1^2+\cdots+X_n^2$ and $X_1,\cdots,X_n$ is a basis for the right-invariant Lie algebra of $G$. It is also shown that the closure of $dT(\Delta+X)$ (where $X$ is in the Lie algebra) is a semi-group infinitesimal generator. Some examples show that if $L^+=L$ is not elliptic, then $dU(L)$ need not be essentially self-adjoint.

*I. Cuculescu, C. Foiaş* (Bucharest)

908:

**Horne, J. G., Jr. Some multiplicative aspects of ideal structure theory.** Fund. Math. **48** (1959), 27–55.

Willcox [Pacific J. Math. **6** (1956), 177–192; MR **18**,

53] has investigated Banach algebras $B$ which he called G.S. algebras, whose defining properties are as follows: (1) the space $S(B)$ of regular maximal ideals in the Stone topology is a Hausdorff space; and (2) every point in $S(B)$ is contained in an open set whose closure has regular kernel. Of particular interest are the G.S. algebras which are strongly semi-simple (s.s.s.). These properties make sense for arbitrary rings; the study of ideals in a s.s.s. G.S. ring $R$ is the main topic of the paper. An ideal $I$ of $R$ is called an $O$-ideal if, for each $f \in I$, there exists $e \in I$ such that $fOe$ (i.e., $ex=xe=x$ for all $x$ in the principal ideal generated by $f$). Unless $R$ is commutative this is a different notion of an $O$-ideal than that considered previously by the author [Trans. Amer. Math. Soc. **90** (1959), 408–430; MR **21** #7429], but this definition is more useful here. The set $M(R)$ of maximal $O$-ideals of $R$ is examined in various topologies including one which makes $M(R)$ a homeomorph of $S(R)$. It is shown that if $R_1$, $R_2$ are s.s.s. G.S. rings which are isomorphic as multiplicative semi-groups then their structure spaces $S(R_k)$ are homeomorphic. This generalizes a conclusion of Civin and Yood [Duke Math. J. **23** (1956), 325–334; MR **17**, 1227] for the special case where $R_k$ is a regular Banach algebra. Dual $O$-ideals are introduced and used to show that, under a multiplicative semi-group isomorphism of $R_1$ onto $R_2$, the image of a regular maximal ideal is another such ideal. Also treated are prime-like and primary ideals. A primary ideal in a s.s.s. G.S. ring is also a s.s.s. G.S. ring. This work generalizes some results of Kohls [Fund. Math. **45** (1957), 28–50; MR **21** #1517]. The various notions of the paper are investigated in settings of maximum generality before being applied to the case of s.s.s. G.S. rings.

*B. Yood* (Eugene, Ore.)

909:

**Ionescu Tulcea, C.; Simon, Arthur B. Spectral representations and unbounded convolution operators.** Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1765–1767.

The first part of the paper is devoted to a generalization of the spectral representation theorem for semi-groups of normal operators proved by A. E. Nussbaum [Ann. of Math. (2) **69** (1959), 133–141; MR **21** #305] and C. Ionescu Tulcea [J. Math. Mech. **8** (1959), 95–109; MR **21** #306]. Suppose $S$ is a locally compact space and $M(S)$ the vector space of complex Radon measures on $S$ with compact support. Further suppose that $A \subseteq M(S)$ is a vector subspace which under some multiplication, $(\nu, \mu) \to \nu\mu$, is an algebra. A continuous complex valued function $\chi$, defined on $S$, is a character of $\{S, A\}$ if $\chi \neq 0$ and $\int_S \chi d\nu \int_S \chi d\mu = \int_S \chi d\nu\mu$ for every $\nu, \mu \in A$. If $r$ is a locally bounded positive function on $S$, set

$$E(r) = \{\chi : |\chi(s)| < r(s), \quad s \in S\}.$$

For any such $r$, let $s \to U_s$ be a weakly continuous mapping of $S$ onto a set of scalar operators on a Banach space $X$ such that $\|U_s\| \leq r(s)$ and $\int_S U_s d\nu \int_S U_s d\mu = \int_S U_s d\nu\mu$ for each $\nu, \mu \in A$. Under suitable conditions on the spectral measure of each $U_s$, there exists a spectral family $\{m_{x,x'}; x \in X, x' \in X'\}$ on $E(r)$ such that

$$\langle U_s x, x' \rangle = \int_{E(r)} \chi(s) dm_{x,x'}(\chi).$$

The second part of the paper is devoted to the extension of the ideas of Fourier transformations and multiplier transformations from the situation where $S$ is a locally compact abelian group to arbitrary locally compact spaces. The extension amounts to giving sufficiently many axioms so that appropriate quantities can be defined which parallel the situation for locally compact groups. A detailed account of all the definitions and axioms would be too involved and lengthy for a review. The authors announce a theorem which generalizes results of R. A. Kunze [Ann. of Math. (2) **69** (1959), 1–14; MR **21** #1491].

*A. Devinatz* (St. Louis, Mo.)

910:

**Guichardet, M. Alain. Sur un problème posé par G. W. Mackey.** C. R. Acad. Sci. Paris **250** (1960), 962–963.

Résumé de l'auteur: "G. W. Mackey pose le problème suivant [Trans. Amer. Math. Soc. **85** (1957), 134–165; MR **19**, 752; p. 163]: une représentation sans multiplicité d'une $*$-algèbre de Banach séparable dans un espace hilbertien séparable peut-elle être décomposée en représentations irréductibles deux à deux non équivalentes? Ce problème reçoit ici une réponse positive."

911:

**Rothe, E. H. A note on gradient mappings.** Proc. Amer. Math. Soc. **10** (1959), 931–935.

In this paper, a Banach space $E$ is said to have the property $P$ if there is a linearly independent system $(f_i)$, fundamental in the conjugate space $E^*$ of $E$, and such that, for some constant $M > 0$, there is a projection of norm at most $M$ on each of the subspaces

$$\{x : f_1(x) = \cdots = f_n(x) = 0\}$$

of $E$. A sufficient condition for this is that $E$ be a reflexive space with a base; more generally, it holds if $E$ has a base $(b_i)$ and the system $(f_i)$ defined by $x = \sum_{i=1}^{\infty} f_i(x) b_i$ $(x \in E)$ is fundamental in $E^*$.

The author then proves the following result. Let $E$ be a (real) Banach space with property $P$. Let $I(x)$ be a scalar function defined in a convex subset $V$ of $E$, having a continuous Fréchet differential $D(x, h)$; and suppose that, given $\eta > 0$, there is a finite set $l_1, l_2, \cdots, l_n$ in $E^*$ such that

$$|I(x+h) - I(x)| \leq \eta \|h\|$$
$$(x \in V, \quad x+h \in V, \quad |l_i(h)| \leq \eta \|h\| \quad (1 \leq i \leq n)).$$

Then the mapping of $V$ into $E^*$ induced by $D(x, h)$ is completely continuous.

The corresponding result for Hilbert space was proved by the author in an earlier paper [Duke Math. J. **15** (1948), 421–431; MR **10**, 548].

*F. Smithies* (Cambridge, England)

912:

**Kaazik, Yu. Ya. [Kaasik, Ü.] Über die Konvergenz von Iterationsmethoden.** Tartu Riikl. Ül. Toimetised **62** (1958), 80–98. (Russian. Estonian and German summaries)

Aus der Zusammenfassung des Autors: "Es sei $P(x)$ ein analytischer Operator, der den Banachschen Raum in den linearen normierten Raum abbildet. Es werden für die Gleichung $P(x)=0$ die Iterationsmethoden

$$\Delta x_{n+1} = x_{n+1} - x_n = G_n \Gamma_n P(x_n) \quad (n = 0, 1, \cdots),$$

betrachtet, wo $x_0$ die Anfangsnäherung, $\Gamma_n = [P'(x_n)]^{-1}$

und $G_n$ ein linearer Operator ist, der von den Operatoren $E, \Gamma_n P'(x_n), \cdots, \Gamma_n P^{(k)}(x_n)$ ($k$ ist eine fixierte positive ganze Zahl) und von dem Element $\Gamma_n P(x_n)$ zusammengestellt ist."

913:

**Kaazik, Yu. Ya. [Kaasik, Ü.]; Tamme, È. È. Über eine Methode zur angenäherten Lösung nichtlinearen Operatorgleichungen.** Tartu Riikl. Ül. Toimetised **62** (1958), 99–116. (Russian. Estonian and German summaries)

Aus der Zusammenfassung der Autoren: "Es sei $P(x)$ ein analytischer Operator, der einen Banachschen Raum in einen linearen Raum abbildet. In der vorliegenden Arbeit werden für die Gleichung $P(x)=0$ die Iterationsverfahren

$$\Delta x_k = x_k - x_0$$

$$= -[E + \sum_{i=2}^{k} \frac{\gamma_i}{i!} \Gamma_0 P^{(i)}(x_0)\Delta x_{k-i+1}^{\alpha} \cdots \Delta x_{k-2}^{\alpha} \Delta x_{k-1}^{\beta}]^{-1}$$

$$\times [\Gamma_0 P(x_0) - \sum_{i=1}^{k} \frac{\gamma_i - 1}{i!} \Gamma_0 P^{(i)}(x_0)\Delta x_{k-i+1}^{\alpha} \cdots \Delta x_{k-2}^{\alpha} \Delta x_{k-1}^{\beta+1}]$$

($x_0$ ist die Anfangsnäherung und $\Gamma_0 = [P'(x_0)]^{-1}$) betrachtet, wo entweder $\alpha=\beta=1$ und $0 \leqq \gamma_i \leqq 2$ oder $\alpha=0, \beta=i-1$ und $0 \leqq \gamma_i \leqq i$."

## CALCULUS OF VARIATIONS

914:

**Osborn, Howard. On the foundations of dynamic programming.** J. Math. Mech. **8** (1959), 867–872.

Sia $X_T$ la classe delle funzioni $x(t)$ definite per $t \in [0, T]$, ivi continue con derivata continua a tratti e tali che $x(0)=0$. Sia $F(x, u)$ una funzione con derivate seconde continue in tutto il piano $(x, u)$ e sia tale che per ogni costante $c$ esista

$$f(c, T) \stackrel{\text{df}}{=} \min_{\substack{x \in X_T \\ x(T)=c}} \int_0^T F(x(t), x'(t))dt.$$

Se $x(t)$ è funzione minimizzante ha interesse per la teoria della programmazione dinamica [R. Bellman, *Dynamic programming*, Princeton Univ. Press, Princeton, N.J., 1957; MR **19**, 820] di approssimare la funzione $p(c, T) \stackrel{\text{df}}{=} x'(T)$ mediante la $F$ ed i valori $f(c_1, T-\varepsilon)$ per un $\varepsilon>0$ fissato ed un certo insieme di valori di $c_1$. L'A. rende rigoroso un metodo euristico usato a questo scopo da R. Bellman e indica altri metodi possibili.

*R. Conti* (Florence)

915:

**Beckert, Herbert. Über ein singuläres mehrdimensionales Variationsproblem.** Math. Ann. **135** (1958), 203–218.

Consider the problem of minimizing the integral

$$\iint_E (a_{11}u_x^2 + 2a_{12}u_x u_y + a_{22}u_y^2 + 2a_{13}u_x u + 2a_{23}u_y u + a_{33}u^2)dxdy$$

in a class of admissible functions $u$ taking on prescribed boundary values along the boundary $S$ of $E$. The author establishes an existence theorem for the case when the problem is singular along a segment $S_0$ of $S$. Apart from continuity and differentiability assumptions, the author assumes that the integrand dominates $u_x^2+u_y^2+u^2$ on every region whose closure is interior to $E$. Moreover, it is assumed that $S_0$ is nowhere tangent to the characteristic of the Euler equations. It is shown that if a segment of $S_0$ is everywhere tangent to the characteristic, the boundary values cannot be prescribed on this segment.

*M. R. Hestenes* (Los Angeles, Calif.)

916:

**Faedo, Sandro. Esistenza dell'estremo assoluto per gli integrali doppi in forma ordinaria.** Rend. Sem. Mat. Fis. Milano **29** (1959), 221–232. (English summary)

In this expository paper the author analyzes recent results concerning existence of the minimum in problems of calculus of variations for double integrals in the line of Tonelli's method. In the particularly difficult situation where usual conditions do not assure compactness, then, among other devices, smoothing processes have been used to deduce, from any minimizing sequence, a compact one. J. Cecconi [Riv. Mat. Univ. Parma **6** (1955), 45–64; MR **19**, 750] and D. Greco [Ricerche Mat. **5** (1956), 159–166; MR **19**, 750] recently proved the existence of the minimum for a positive quasi-regular nonparametric double integral $(D)\int f(x, y, z, z_x, z_y)dxdy$, with prescribed boundary values, essentially under the hypothesis $f \geqq M(z_x^2+z_y^2)+N$, $M, N>0$ constants. J. Cecconi used the smoothing method of the reviewer; D. Greco, an elegant variant of the methods of Lebesgue and Tonelli, also related to the one given by the reviewer. The author discusses these smoothing processes.

*L. Cesari* (Ann Arbor, Mich.)

## GEOMETRY

See also 685.

917:

**Ślebodziński, W. On the definition of geometry.** Wiadom. Mat. (2) **1** (1955/56), 153–162. (Polish)

Une explication élémentaire de la notion "géométrie" à l'aide des groupes de mouvements. *A. Švec* (Prague)

918:

**Avishalom, Dov. Perimeter-bisectors in a triangle passing through middle of side or through vertex.** Riveon Lematematika **13** (1959), 46–49. (Hebrew)

For straight lines bisecting the perimeter of a triangle and passing through the midpoints of its edges the author proves: (1) the three bisectors are concurrent; (2) they are parallel to the three angle-bisectors. Similarly, the three perimeter-bisectors passing through the vertices of a triangle are concurrent at the incenter of the triangle (i.e., the perimeter centroid).

*B. Grünbaum* (Seattle, Wash.)

919:

**Jarden, Dov. Synthetical proof for the theorem on the centre of the perimeter-bisectors in a triangle passing through middle of side.** Riveon Lematematika **13** (1959), 50. (Hebrew)

Theorem 1 of the preceding paper [see preceding review] is derived from Theorem 2 and the following remark: Three (straight) lines through the three vertices of a triangle are concurrent if and only if the three lines, parallel to the first ones and passing through the midpoints of the edges, are concurrent.

*B. Grünbaum* (Seattle, Wash.)

920:

**Ionescu-Bujor, Constantin. Sur certaines suites de transformations.** Bul. Inst. Politehn. Bucureşti **20** (1958), no. 2, 13–22. (Russian, English and German summaries)

Let a point $O$ be given in a triangle $ABC$, and let the lines $OA$, $OB$, $OC$ intersect the sides of the triangle in $A_1$, $B_1$, $C_1$; the triangle $A_1B_1C_1$ is the 'pedal triangle' of $O$. If a point $M$ in the plane is given, and $MA$, $MB$, $MC$ intersect the sides of the pedal triangle in $A'$, $B'$, $C'$, then the lines $A_1A'$, $B_1B'$, $C_1C'$ pass through one point. This point $Q$ is the 'pedal transform' of $M$ in the triangle $ABC$ with respect to $O$. This relationship is expressed in homogeneous coordinates with $T(ABC)$ as fundamental triangle. Another relationship is the 'projective inversion' $P$ given by $\rho\zeta_i = 1/x_i$, $i = 1, 2, 3$, transforming point $M(x_i)$ into $N(\xi_i)$. Some theorems are derived on the transformations $P_q = H^q P H^{-q}$ and especially $C_{q+1} = HP_q$ where $H$ is a homography.

*D. J. Struik* (Cambridge, Mass.)

921:

**Stein, Erwin Walter. Eine Diskriminante des vollständigen Vierecks.** Math. Nachr. **17**, 137–142 (1959).

Un quadrangle de sommets $A$, $B$, $C$, $D$ est dit de première espèce si aucun des sommets n'est intérieur au triangle formé par les trois autres, de deuxième espèce dans le cas contraire; il est dit dégénéré si au moins trois sommets sont alignés. En désignant par $a$, $b$, $c$, $d$, $e$, $f$ les longueurs des divers segments joignant deux sommets, l'auteur forme un discriminant $H(a, b, c, d, e, f)$ qui est positif, négatif ou nul suivant que le quadrangle est de première espèce, de deuxième espèce ou dégénéré. Il énonce ensuite quelques propositions relatives aux variations de $H$ lorsque sont fixés certains éléments du quadrangle.

*M. Decuyper* (Lille)

922:

**Lietzmann, W. ★Experimentelle Geometrie.** B. G. Teubner Verlagsgesellschaft, Stuttgart, 1959. 111 pp. DM 12.60.

A lifetime of accumulated knowledge and didactic experience must have gone into this valuable book, written in simple and clear language and avoiding professional or too specialized terminology. Its main merit, in these times of reforming mathematics curricula in high school, is its style, as the author repeatedly points to and opens up vistas to problems solved or attacked only in the 20th century. Almost every page brings a stimulating fact or geometric aperçu, flavored by historic remarks and whetting the appetite of the reader to consult the freely given source material. We hope that this book will soon appear in an English translation, so that it will be available to all ambitious high school teachers or pupils.

The title "Experimental Geometry" means a geometry where construction is carried out by hand and not just "with the tongue", as Jacob Steiner expressed it; it is an approximation mathematics and not a precision mathematics (an expression F. Klein coined). Within the framework of its title the book covers quite diverse fields: constructions with compass and ruler and with a whole host of other instruments (like an isoceles rectangular triangle; a carpenter's square; a triangulation compass; etc.); a discussion of the limitations in accuracy these instruments involve; methods of judging numerically the "simplicity" of the procedure, when different tools and methods are used to arrive at the same result; constructions and solutions obtained by the folding of paper (our interest being lately stimulated by the Japanese and their skill), which are especially useful in proving the Pythagorean theorem by several methods. Moreover, we find in this little book construction problems solved by reflection tools, for the accurate construction of tangents to curves; reflection considerations lead to philosophical questions of right and left, screw sense, etc.; a scissor geometry, capable of solving area problems (cutting up and assembling), which opens up the field of topology (Moebius strip, etc.); a string geometry for constructing conic sections and solving minimum area problems, etc.; the theory of knots, area, probability, acoustics and even more fields, which lend themselves to empirical, geometric solutions.

*S. R. Struik* (Cambridge, Mass.)

923:

**Manevič, V. A. On a method in descriptive geometry.** Mat. Sb. (N.S.) **48 (90)** (1959), 105–116. (Russian)

924:

**Skof, Fulvia. Sulla costruzione grafica del piano osculatore ad una quartica gobba di prima specie.** Ist. Lombardo Accad. Sci. Lett. Rend. A **93** (1959), 175–198.

The author gives a very clear exposition of results of H. Drasch, F. Machovec and U. Cassina concerning osculating planes to quartic curves of the first kind, and applies these results to the effective construction of such planes by methods of descriptive geometry.

*A. Gutwirth* (Berkeley, Calif.)

925:

**Barsotti, Leo. Considerations on the symmetry of curves defined by polar equations.** Soc. Parana. Mat. Anuário (2) **1** (1958), 28–31. (Portuguese. French summary)

926:

**Alardin, Félix. Sur les points doubles des transformations conformes de $E^3$.** Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 369–386.

Werden die kartesischen Koordinaten eines Raumpunktes zu einer rein vektoriellen Quaternion $z$ zusammengefaßt, so lassen sich die konformen Transformationen des dreidimensionalen euklidischen Raumes nach E. Study durch eine Quaternionen-projektivität $z' = (mz+n)(pz+q)^{-1}$ beschreiben, vorausgesetzt daß diese

den vektoriellen Charakter von $z$ bewahrt, was den Koeffizienten fünf homogene Bedingungen auferlegt. Schließt man den durch $p=0$ gekennzeichneten trivialen Fall der bloßen Ähnlichkeiten aus, dann kann man die Gleichung auf die Form $z'=\alpha+\lambda p(z+\beta)^{-1}p^{-1}$ bringen, wobei $\alpha=mp^{-1}$ und $\beta=p^{-1}q$ rein vektoriell sind und $\lambda$ einen nichtverschwindenden Skalar bezeichnet. Aus dieser Darstellung ist unmittelbar der Aufbau der Transformation $z\to z'$ aus einer Schiebung, Inversion, Drehung, Streckung und einer nochmaligen Schiebung zu ersehen.— Die Fixpunkte der Transformation sind durch die rein vektoriellen Lösungen der Gleichung $z'=z$ gegeben, und deren Bestimmung läuft nach der Substitution $y=p(z+\beta)$ auf die Auflösung der quadratischen Quaternionengleichung $y^2=by+c$ hinaus, in welcher zur Abkürzung $p(x+\beta)=b$ und $\lambda p^2=c$ gesetzt wurde. Die tatsächliche Auflösung dieser Gleichung, die zahlreiche Fallunterscheidungen erfordert, wird (unter Beschränkung auf reelle Lösungen) mit aller Sorgfalt durchgeführt. Sie führt zur Klassifikation der konformen Raumtransformationen hinsichtlich ihrer reellen Doppelpunkte, die im allgemeinen isoliert (in den Anzahlen 2, 1 oder 0) auftreten, jedoch auch eine Kreislinie oder eine Kugelfläche erfüllen können. *W. Wunderlich* (Zbl 77, 341)

927:

**Świerczkowski, S. On chains of regular tetrahedra.** Colloq. Math. **7** (1959), 9–10.

The author considers the group generated by reflections in the face-planes of a regular tetrahedron in Euclidean space. Since the dihedral angle arcsec 3 is incommensurable with $\pi$, this group is not discrete. The author proves that it contains no translation (except the identity).

*H. S. M. Coxeter* (Toronto)

928:

**Kuiper, N. H. ★Analytische meetkunde (verklaard met lineaire algebra). [Analytic geometry (interpreted by linear algebra).]** N. V. Noord-Hollandsche Uitgeversmaatschappij, Amsterdam, 1959. viii + 274 pp. Guilders 17,50; $4.75.

This is no ordinary textbook. Right from the beginning it defines vectors and $n$-vector space in modern symbolism, gives in concise form the axioms to be utilized throughout, treats the different topics (e.g., affine plane, classification of endomorphisms, quadratic and symmetric bilinear functions, some applications to statistics, motions and affine transformations, and some topology) by up-to-date methods and thus creates a model of a book for the budding research-scientist, ingenious, clear, consistent in structure. Definitions are chosen so as to be fit for generalizations. It is helpful that an asterisk indicates a subject or a problem to be difficult, and to be restudied after some experience. Significant for our times is its planned use for the ambitious Dutch high school teacher, acquiring up-to-date knowledge in this field, to be tested in additional examinations.

*S. R. Struik* (Belmont, Mass.)

929:

**Pickert, Günter. Gemeinsame Kennzeichnung zweier projektiver Ebenen der Ordnung 9.** Abh. Math. Sem. Univ. Hamburg **23** (1959), 69–74.

In an earlier paper [Canad. J. Math. **9** (1957), 389–399; MR **19**, 445] Ostrom has shown: if $P$, $Q$ are distinct points, and $p$, $q$ are distinct lines, of a projective plane $\pi$, such that there are no incidences between $P$, $Q$ and $p$, $q$, or between $pq$ and $PQ$, and if $\pi$ is $(P, p)$ and $(Q, q)$ transitive, then $\pi$ is Desarguesian. This result is false, and Ostrom later proved it to be true except possibly when $\pi$ has order nine [Canad. J. Math. **10** (1958), 507–512; MR **20** #6065]. In this note, the author sharpens the result by showing that Ostrom's original theorem is true excepting exactly when $\pi$ is the plane coordinatized by the proper nearfield of order nine, or is the dual of this last plane. (And there exists at least one more non-Desarguesian plane of order nine.) *D. R. Hughes* (Ann Arbor, Mich.)

930:

**Hughes, D. R. Collineation groups of non-Desarguesian planes. I. The Hall Veblen-Wedderburn systems.** Amer. J. Math. **81** (1959), 921–938.

This paper is devoted to the study of the collineation groups of a class of non-desarguesian planes. Let $K\neq \mathrm{GF}(2)$ be an arbitrary field over which there exist irreducible quadratics. Let $f(z)=z^2-pz-q$ be such an irreducible quadratic over $K$. Let $R$ be the set of all elements $\lambda a+b$, with $a$ and $b$ in $K$ and addition and multiplication defined in $R$ by

$$(1)\quad (\lambda a+b)+(\lambda c+d) = \lambda(a+c)+(b+d),$$

$$(2)\quad (\lambda\cdot 0+b)(\lambda c+d) = \lambda(bc)+(bd),$$

$$(3)\quad (\lambda a+b)(\lambda c+d) = \lambda(ad-bc+pc) + bd-a^{-1}c(b^2-pb-q),\quad a\neq 0.$$

Such a system is called a Hall V-W system and the projective plane $\pi$ appropriately coordinatized by it is called a Hall V-W plane. Section 2 describes properties of $R$. Section 3 exhibits various collineations of $\pi$. These are entitled the translation group $\mathfrak{T}$, the automorphism group $\mathfrak{A}$, the multiplicative group $\mathfrak{M}$, the autotopism group $\mathfrak{S}$, the linear group $\mathfrak{L}$, and the involution $\delta$. Section 4 examines the possibility of the existence of collineations other than those generated by the groups of Section 3. Let $\mathfrak{B}=\{\mathfrak{L}, \delta\}$ = group generated by $\mathfrak{L}$ and $\delta$ and let $\mathfrak{G}$ designate the group of all collineations of $\pi$. One of the main results in this section asserts that if $K$ is finite and if $K\neq \mathrm{GF}(3)$, then $\mathfrak{G}=\mathfrak{B}\mathfrak{S}\mathfrak{A}\mathfrak{T}$, and both $\mathfrak{S}$ and $\mathfrak{L}$ are normal in $\mathfrak{G}/\mathfrak{T}$. Section 5 investigates the conditions under which non-isomorphic Hall V-W systems, over the same center $K$, lead to isomorphic planes. The main result here asserts the following: If $f(z)$ and $f_1(z)$ are irreducible quadratics over $K$, then the Hall V-W planes defined by $f(z)$ and $f_1(z)$ are isomorphic provided $f(z)$ and $f_1(z)$ have zeros in the same quadratic extension field of $K$. Hence two finite Hall V-W systems of the same order define isomorphic projective planes. The concluding Section 6 restricts $K$ to GF(3). The group $\mathfrak{G}$ is again non-solvable in this case.

*H. J. Ryser* (Columbus, Ohio)

931:

**Hughes, D. R. Collineation groups of non-Desarguesian planes. II. Some seminuclear division algebras.** Amer. J. Math. **82** (1960), 113–119.

The author considers a class of non-associative division algebras and the projective planes coordinatized by them. The principal objective is the determination of the collineation groups of the planes. These groups turn out to be

solvable like those associated with the twisted fields of Albert. This contrasts sharply with the state of affairs for finite Hall V-W and Hughes planes.

Let $F$ be a field and let $\sigma$ be a non-identity automorphism of $F$. Let $\delta_0$ and $\delta_1$ be elements of $F$ such that

$$\delta_0 \neq w^{1+\sigma} + \delta_1 w$$

for any $w$ in $F$. Let $R$ be a two-dimensional vector space over $F$ with basis elements 1 and $\lambda$. Define multiplication in $R$ by

$$(x+\lambda y)(u+\lambda v) = (xu+\delta_0 y^\sigma v)+\lambda(yu+x^\sigma v+\delta_1 y^\sigma v).$$

$R$ is a division ring and the central theorem of the paper asserts the following. If $R$ is finite then the autotopism group $\mathfrak{G}$ of $R$ is solvable and $\mathfrak{G}$ has a sub-normal series

$$\mathfrak{G} \supseteq \overline{\mathfrak{G}} \supseteq \mathfrak{H}_1 \supseteq \mathfrak{H} \supseteq 1.$$

Here $\mathfrak{G}/\overline{\mathfrak{G}}$ has order one or two and has order one if $\sigma^2 \neq 1$; $\overline{\mathfrak{G}}/\mathfrak{H}_1$ is isomorphic to a subgroup of the automorphism group of $F$; $\mathfrak{H}_1/\mathfrak{H}$ is isomorphic to a subgroup of the multiplicative group of $F$; $\mathfrak{H}$ is isomorphic to the multiplicative group of the left nucleus of $R$. This theorem implies as a corollary that if $\pi$ is the projective plane coordinatized by the finite semi-nuclear division algebra $R$, then the collineation group of $\pi$ is solvable.

*H. J. Ryser* (Columbus, Ohio)

932:

**Neumann, M. Sur la représentation de Poincaré de la géométrie de Lobacevski.** Lucrăr. Şti. Inst. Ped. Timişoara. Mat.-Fiz. **1958**, 83–102 (1959). (Romanian. French and Russian summaries)

L'A. étudie la géométrie de Lobačevskiĭ en représentant dans le modèle de Poincaré les points par les involutions elliptiques sur le bord du demi-plan; à l'involution hyperbolique on attache la droite hyperbolique qui passe par ses points doubles.

*A. Švec* (Prague)

## CONVEX SETS AND GEOMETRIC INEQUALITIES

See also B1459.

933:

**Makai, E. Steiner type inequalities in plane geometry.** Period. Polytech. Elec. Engrg. **3** (1959), 345–355.

Let $S$ be a closed point set having a well-defined perimeter $L$ and a well-defined area. Let $r$ be the radius of the greatest inscribed circle. Let $L_\rho$ be the perimeter of an "outer parallel point set", that is, the union of all circular disks of radius $\rho$ whose centers are points of $S$. Let $L_{-\rho}$ be the perimeter of an "inner parallel point set", that is, the closure of the set of centers of all those circular disks of radius $\rho$ which lie entirely in the interior of $S$. The author proves that, if $S$ is simply connected,

$$L_\rho \leqq L+2\pi\rho \quad (\rho > 0), \qquad L_{-\rho} \leqq L-2\pi\rho \quad (\rho < r).$$

Moreover, if $S$ is $k$-tuply connected (having $k-1$ holes), the former inequality remains valid whereas the latter has to be replaced by $L_{-\rho} \leqq L+2(k-2)\pi\rho$.

*H. S. M. Coxeter* (Toronto)

934:

**Fejes Tóth, L. Über Extremaleigenschaften der regulären polyeder.** Bŭlgar. Akad. Nauk Izv. Mat. Inst. **4**, no. 1, 121–130 (1959). (Bulgarian and Russian summaries)

The author considers a convex polyhedron having $e$ vertices, $k$ edges, $f$ faces, circumradius $R$, inradius $r$, and volume $V$. He writes $p=2k/f$, $q=2k/e$, so that $p$ is the average number of sides of a face and $q$ is the average number of edges at a vertex. He combines his inequality

$$k \sin\frac{2\pi}{p}\left(\tan^2\frac{\pi}{p}\tan^2\frac{\pi}{q}-1\right)r^3 \leqq 3V$$

[Canad. J. Math. **2** (1950), 22–31; MR **11**, 386; p. 24] with Florian's

$$3V \leqq 2k\cos^2\frac{\pi}{p}\cot\frac{\pi}{q}\left(1-\cot^2\frac{\pi}{p}\cot^2\frac{\pi}{q}\right)R^3$$

[Monatsh. Math. **60** (1956), 130–156; MR **17**, 1235] to obtain a new derivation of his

$$\frac{R}{r} \geqq \tan\frac{\pi}{p}\tan\frac{\pi}{q}$$

[Acta Math. Acad. Sci. Hungar. **7** (1956), 31–48; MR **18**, 63; p. 42]. His conjecture, that of all convex polyhedra containing a given sphere the cube has the smallest sum of edge-lengths, has been established by A. S. Besicovitch and H. G. Eggleston [Quart. J. Math. Oxford Ser. (2) **8** (1957), 172–190; MR **20** #1950].

*H. S. M. Coxeter* (Toronto)

935:

**Menger, Karl; Schweizer, Berthold; Sklar, Abe. On probabilistic metrics and numerical metrics with probability 1.** Czechoslovak Math. J. **9** (**84**) (1959), 459–466. (Russian summary)

The authors criticize A. Špaček's definition of random metric [same J. **6** (**81**) (1956), 72–74; MR **18**, 330] as being too restrictive for applications on the following grounds: (1) if, for each choice of $a$ and $b$, the distance $d(a, b)$ between $a$ and $b$ is independent of the distance between $b$ and $a$, then $d(a, b)=d(b, a)=D(a, b)$ with probability one, where $D(a, b)$ is some fixed classical metric; (2) if $d_1=d(a, b)$, $d_2=d(b, c)$, $d_3=d(c, a)$ are independent, then ess sup $(d_1+d_2+d_3) \leqq 2$ ess inf $(d_1+d_2+d_3) < +\infty$, and, if ess inf $d_1=0$, then both $d_2$ and $d_3$ are constant $(>0)$.

*H. P. McKean, Jr.* (Cambridge, Mass.)

## DIFFERENTIAL GEOMETRY

See also 737, 739, 826, 971, 993.

936:

**Su, Buchin. ★The general projective theory of curves.** Science Press, Peking, 1958. iv+242 pp.

This volume, although devoted to a branch of geometry on which excellent treatises exist, contains many new features based on the use and developments of notions introduced by this reviewer.

Ch. I is concerned with the theory of plane curves, including the theory of contact at a regular point. Ch. II deals with singularities (inflexional points of any order and cusps) of plane curves (Bompiani's osculants);

applications are made to the study of plane sections of the developable surface determined by a skew curve. These geometric notions allow (Ch. III) a new treatment of the projective geometry of skew curves (geometric determination of tetrahedrons of reference attached to a point; canonical developments). Ch. IV gives a broad account of the intersection invariants of two skew curves and their consequences for holonomic and non-holonomic surfaces. Ch. V extends to curves in a four-dimensional space the results of Ch. III for skew curves using the same geometric tools; a further extension to curves in $n$-dimensional spaces is attempted in Ch. VI by the use of Wronskians.

A substantial bibliography completes the volume.

*E. Bompiani* (Rome)

937:

**Anisimova, E. P. A new complex of cubic space curves.** Moskov. Oblast. Pedagog. Inst. Uč. Zap. **57** (1957), 165–172. (Russian)

This complex consists of the cubic space curves which pass through the three points $P$, $Q$, $M$, intersect a given line $l$ twice and possess a chord passing through $P$. Also studied are the $\infty^2$ congruences of complex curves which possess a common chord passing through $M$. This paper is therefore related to the studies of J. de Vries [Proc. Akad. Wetensch. Amsterdam **37** (1934), 129–132] and of V. C. Morton and M. T. Chapple [Quart. J. Math. Oxford Ser. **19** (1948), 133–139; MR **10**, 207].

*D. J. Struik* (Cambridge, Mass.)

938:

**Reiche, Erhard. Liniengeometrische Untersuchungen mit Geodätischen. I, II.** Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 347–396.

Vorwort des Verfassers: "Den Untersuchungen wird eine zweidimensionale stetige Mannigfaltigkeit von Geodätischen oder auch zusammenhängenden Stücken solcher Linien auf einer Fläche einer zweidimensionalen Riemannschen Geometrie zugrundegelegt. Die verschiedenartigen und in vielen Fällen komplizierten Nachbarschaftsverhältnisse zweier infinitesimal benachbarter Geodätischer $u$, $v$ und $u+du$, $v+dv$ ($u$, $v$ sind Linienkoordinaten der Geodätischen) lassen wohl im allgemeinen (wenn man einschneidende Beschränkungen in der Auswahl und Länge solcher Linien vermeiden will) eine eindeutige Bestimmung eines skalaren gegenüber Koordinatentransformationen invarianten Differentials $ds$ nicht zu, das diesen Linien als Masszahl zugeordnet werden könnte.

Die vorliegende Arbeit führt anstelle eines skalaren $ds$ einen von $u$, $v$, $du$, $dv$ abhängenden invarianten 'Linienvektor', das 'Fahrelement' $d\mathfrak{s}$ als metrische Grundgrösse ein; $d\mathfrak{s}$ liefert den metrischen Zusammenhang der beiden benachbarten Geodätischen längs ihrer ganzen Erstreckung. Dabei ist es im allgemeinen auch nicht auf dem Wege über eine Betragsbildung von $d\mathfrak{s}$ möglich, zu einem eindeutig bestimmten skalaren $ds$ zu gelangen; die hier benutzten Linienvektoren und damit auch $d\mathfrak{s}$ besitzen nämlich im allgemeinen nicht einen eindeutig bestimmten 'Betrag'.

Auf einfache Weise kommt man zur 'Liniendichte', einem Gegenstück zum Flächenelement der Differentialgeometrie. Es ist eine Invariante, die von Poincaré gefunden und die in Blaschkes Abhandlungen zur Integralgeometrie näher untersucht worden ist."

*T. Takasu* (Yokohama)

939:

**Bourion, Georges. Sur une présentation de la théorie des surfaces minima.** Bul. Inst. Politehn. Iași (N.S.) **4** (8) (1958), no. 3–4, 75–76. (Russian and Romanian summaries)

Defining a minimal surface $\mathbf{x}=\mathbf{x}(\alpha, \beta)$ as a regular $C^2$-surface on which the vector product $\mathbf{n} \wedge \mathbf{dx}$ is a complete differential, the author derives in a direct fashion well-known properties of minimal surfaces in the small: local introduction of isothermic parameters, theorem of Weierstrass, analytic character, etc. (See T. Radó, *On the problem of Plateau* [Springer, Berlin, 1933; Chap. II].)

*J. C. C. Nitsche* (Minneapolis, Minn.)

940:

**Lumiste, Yu. G. [Lumiste, Ü.] Die $n$-dimensionalen Minimalflächen mit einer $(n-1)$-dimensionalen asymptotischen Richtung in jedem Punkte.** Tartu Riikl. Ül. Toimetised **62** (1958), 117–141. (Russian. Estonian and German summaries)

Aus der Zusammenfassung des Autors: "Es wird die Klassifikation dieser Flächen gegeben und die geometrischen Eigenschaften der Fläche in einzelnen Fällen untersucht."

941:

**Bal, Lascu. Propriétés de certaines surfaces spéciales $h(z)=f(x)+g(y)$.** Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **9** (1958), 39–44. (Romanian. Russian and French summaries)

L'A. détermine les surfaces développables du type (*) $h(z)=f(x)+g(y)$. Il examine quelques réseaux hexagonaux, les familles de ces réseaux étant les intersections des plans de coordonnées avec la surface et leurs trajectoires orthogonales; la condition nécessaire et suffisante pour que les courbes $x=c_1$, $y=c_2$, $z=c_3$ de la surface $z=f(x, y)$ forment un réseau hexagonal est (*).

*A. Švec* (Prague)

942:

**Trušin, G. I. $\lambda$-conjugacy of three directions on $p$-dimensional manifolds in an $n$-dimensional projective space.** Dokl. Akad. Nauk SSSR **129** (1959), 37–39. (Russian)

La conjugaison de deux tangentes en un point d'une variété se définit par rapport aux formes asymptotiques quadratiques; par extension, trois tangentes sont dites $\lambda$-conjuguées si elles annulent les formes trilinéaires adjointes aux formes asymptotiques cubiques; dans le cas le plus général, il n'existe pas de directions $\lambda$-conjuguées. Trois familles de courbes sur une variété de dimension $p$ sont dites $\lambda$-conjuguées si leurs tangentes en tout point de la variété sont $\lambda$-conjuguées. Un système $\lambda$-conjugué de dimension $p$ est une variété de dimension $p$ sur laquelle existe un réseau régulier de $p$ familles de courbes telles que trois quelconques d'entre elles soient conjuguées; l'auteur démontre l'existence de systèmes $\lambda$-conjugués de dimension trois; ils dépendent de 12 fonctions arbitraires de trois paramètres. Les notations utilisées sont celles de S. Finikoff.

*M. Decuyper* (Lille)

943:

**Zajtz, A. Sur une hypothèse de M. Biernacki.** Ann. Polon. Math. **7** (1959), 9–11.

Given a closed surface $S$ and a point $O \notin S$. Let $Q$ be the area of the spherical image of $S$ and let $P$ be the area of the projection of $S$ from $O$ onto the unit sphere about $O$. In either mapping each point of the sphere is counted with a multiplicity equal to the number of its original points on $S$. The author gives two examples which disprove Biernacki's conjecture that $P \leq Q$ [cf. Biernacki, Mathematica, Timişoara **23** (1948), 54–59; MR **10**, 186].

*P. Scherk* (Toronto)

944:

**Benson, Donald C. On surfaces with the representation $z=f(x, y)$.** Proc. Amer. Math. Soc. **10** (1959), 980–982.

Étant donné un ensemble ouvert $G$ dans un espace euclidien de dimension $n$ et sa frontière $S$, l'auteur introduit la notion de pseudo-normale en un point. Si en tout point de $S$ existe une pseudo-normale, $S$ est dite "pseudo-smooth". L'auteur établit ce théorème: "Si $S$, frontière d'un ensemble ouvert $G$, est un sous-ensemble 'pseudo-smooth' de l'espace euclidien de dimension $n$ avec un système $N$ de pseudo-normales contenu dans un hémisphère ouvert de la sphère unitaire $S^{n-1}$, il existe alors une direction $l$ telle que toute droite $l'$ parallèle à $l$ coupe $S$ en un point au plus." Il caractérise ensuite, par le diamètre de $N$, un sous-ensemble $N$ contenu dans un hémisphère ouvert.

Les résultats donnent en particulier des conditions pour qu'une surface $S$ admette, dans un système convenable de coordonnées cartésiennes, une représentation $z=f(x, y)$, où $f$ est une fonction à une seule détermination.

*M. Decuyper* (Lille)

945:

**Ryžkov, V. V. Tangential deformation of surfaces.** Mat. Sb. (N.S.) **47 (89)** (1959), 55–110. (Russian)

Étude systématique des déformations tangentielles (t.) des variétés plongées dans l'espace afin ou euclidien; voir les travaux précédents de l'A. [Dokl. Akad. Nauk SSSR **75** (1950), 503–506; **111** (1956), 763–765; Uspehi Mat. Nauk **12** (1957), no. 3 (75), 195–200; MR **13**, 776; **19**, 574, 878]. Seulement les variétés à système complet conjugué peuvent admettre des déformations t. qui ne sont pas ponctuellement-tangentielles (p.-t.). On étudie en détail la déformation t. d'ordre 1 des surfaces à réseau conjugué et des systèmes $n$-conjugués, la déformation t. et p.-t. d'ordre 2 des surfaces d'$A_4$ et la déformation t. métrique des surfaces.

*A. Švec* (Prague)

946:

**Godement, Roger. Variétés différentiables. Résumé des leçons.** Textos de Matemática, No. 2. Instituto de Física e Matemática, Universidade do Recife, 1959. v+51 pp. (polycopiées)

This résumé of lectures given by the author at the University of Recife (1956) is a very lucid exposition of the fundamental concepts on differentiable manifolds which contains the following topics: 1. Exterior algebra; 2. differentiable functions; 3. differentiable manifolds; 4. tangent vectors and differentials; 5. fiber spaces. Although of certain subjects little more than the definition is given, it would be difficult to say more in so small a number of pages.

*L. A. Santaló* (Buenos Aires)

947:

**Lemleĭn, V. G. Local centro-projective spaces of a differentiable manifold and the object $\gamma_{\mu}{}^{\rho}$ which defines an invariant differentiation in a fractional linear group.** Dokl. Akad. Nauk SSSR **129** (1959), 254–256. (Russian)

A tout point $M$ d'une variété différentiable $(V^n)$ de dimension $n$, on associe un espace centro-projectif $(P^n)$ de même dimension en précisant la loi de transformation des coordonnées $u^i$ dans $(P^n)$ qui est associée à la transformation générale des coordonnées locales $x^i$ de $(V^n)$. Cette loi est telle que au produit de deux changements de coordonnées locales de $(V^n)$ est associé le produit des transformations centro-projectives associées. L'auteur démontre que les dérivées du jacobien de la transformation centro-projective induite dans $(P^n)$ coïncident au centre de $(P^n)$ avec les dérivées du jacobien de la transformation des coordonnées locales de $(V^n)$. Il définit ensuite une différentiation invariante dans le groupe fractionnaire linéaire et montre que cette différentiation coïncide en un point donné avec la différentiation ordinaire dans certains systèmes de coordonnées locales.

*M. Decuyper* (Lille)

948:

**Aczél, János. The theory of geometric objects. I, II.** Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **8** (1958), 41–65; 211–243. (Hungarian)

By elementary methods it is shown that in a one-dimensional space there exist no one-dimensional one-component proper differential geometric objects of class higher than 3, and all such objects of classes 1, 2, and 3 are found.

In the proof it is assumed with regard to the function $f$ determining the transformation $\bar{x}=f(x, \alpha_1, \cdots, \alpha_n)$ of the object, only that (1) $f$ is continuous in the component of the object $x$ and the highest derivative $\alpha_n$, and (2) for any $x$ in its domain of definition and arbitrary values $\alpha_1 \neq 0$, $\alpha_2, \cdots, \alpha_{n-1}, f$ is actually dependent on $\alpha_n$; whereas in the previous known proofs of the above-mentioned theorems the function $f$ was assumed either analytic (Ya. S. Dubnov, Yu. E. Penzov) or once continuously differentiable (S. Gołąb).

*Yu. E. Penzov* (RŽMat. **1958** #676)

949:

**Tabata, Fujio. The local infinitesimal deformations of a Riemannian manifold. I.** Tensor (N.S.) **9** (1959), 143–161.

The paper is concerned with infinitesimal deformation problems of a type studied by various authors in the thirties. The structure of a Riemannian space being determined by the metric tensor $g_{ij}(x)$, the author considers the metric tensor as depending also on a parameter $t$, and he determines the infinitesimal deformations in terms of the values of $g$ and of its derivatives with respect to $t$, all evaluated for a particular value of $t$. He takes a more general standpoint than that of previous authors in that he considers a group $G$ of transformations of the form

$$\bar{t} = at+b, \quad \bar{x}^i = \bar{x}^i(x), \quad \bar{g}_{ij} = c^2 g_{ij}.$$

An invariant theory with respect to this group is called a $G$-theory, and objects which are invariant for the group are called $G$-objects. Extensive contacts are indicated between this theory and that of the Lie derivative as given in Schouten [*Ricci-calculus*, Springer, Berlin, 1954; MR **16**, 521].

*E. T. Davies* (Southampton)

950:

**Günther, Paul. Potentialtheoretische Charakterisierung der Kugeln in harmonischen Riemannschen Räumen.** Math. Nachr. **17**, 308–317 (1959).

The reviewer used potential theory to obtain a characterization of harmonic Riemannian spaces [J. London Math. Soc. **25** (1950), 54–57; MR **11**, 436]. In this paper the author uses potential theory to characterize in a Riemannian space of positive definite metric, harmonic at $P$, the geodesic spheres centred at $P$. By considering Green's functions of the first and second kind, $G^{\mathrm{I}}(P, Q)$ and $G^{\mathrm{II}}(P, Q)$, he shows that geodesic spheres centred at $P$ can be characterized (i) by the condition that the normal derivative of $G^{\mathrm{I}}(P, Q)$ is constant as $Q$ varies over the sphere, or (ii) by the condition that $G^{\mathrm{II}}(P, Q)$ is constant as $Q$ varies over the sphere. He shows that in a completely harmonic space, a geodesic sphere centred at $P$ is characterized by the condition that the volume potential $U(Q)$ is a function only of the geodesic distance from $Q$ to $P$.

*T. J. Willmore* (Liverpool)

951:

**Hsu, Chen-Jung. A remark on locally flat infinitesimal connections.** Tôhoku Math. J. (2) **11** (1959), 425–429.

Let $P$ be a differentiable principal fibre bundle over $B$ with group $G$. It is well known that a connection in $P$ is locally flat (i.e., the curvature 0) if and only if the distribution $x \to Q_x$, where $Q_x$ is the horizontal space at $x \in P$, is involutive. Consider a system of submanifolds $B'$ of $B$ such that there is a unique $B'$ through each point of $B$. The author generalizes the above theorem to the induced connections in $P|B'$. *S. Kobayashi* (Vancouver, B.C.)

952:

**Hangan, T. Sur les connexions projectives.** Rev. Math. Pures Appl. **3** (1958), 265–276.

Starting with an $n$-dimensional manifold the author considers a fiber bundle over it whose fibre is projective $n$-space and whose group is the projective group. He defines a connexion on such a bundle in the manner of Ehresmann and then, in traditional fashion, derives a large number of formulas of the usual type. There are no theorems. *W. Ambrose* (Cambridge, Mass.)

953:

**Haimovici, Adolf. Sopra alcuni spazi a connessione affine globalmente equivalenti allo spazio affine.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **26** (1959), 363–367.

A space $L_n$ with affine connection is said to be equivalent in the large to an affine space $A_n$, if there exists a twice differentiable homeomorphism between $L_n$ and $A_n$, which conserves parallelism. Such a space $L_n$ must, of course, be a locally euclidean (or locally flat) space, i.e., the curvature and torsion in it are equal to zero at every point. G. Vrănceanu studied the space with locally euclidean affine connection equivalent in the large to an affine space under the hypothesis that in a certain coordinate system the coefficients of connection $\Gamma_{ij}^{k}$ are constants [Publ. Math. Debrecen **4** (1956), 359–361; Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 5–20; C. R. Acad. Sci. Paris **234** (1956), 1997–1999; MR **18**, 822; **21** #896; **19**, 61]. P. Mocanu [Com. Acad. R. P. Române **2** (1952), 389–395; MR **17**, 407] and T. Postelnicu [Acad. R. P. Romîne Stud. Cerc. Mat. **8** (1957), 279–301; MR **21** #2269] worked also on the same subject. In this paper, the same subject is treated in a more general case with fewer restrictions than that of Vrănceanu, by making use of the recent result of A. Ostrowski [C. R. Acad. Sci. Paris **247** (1958), 172–175; MR **21** #2717] on the univalence of the transformations in an $R^n$. That is, the author obtains a sufficient condition in order that $L_n$ be equivalent in the large to $A_n$, when the coefficients of the connection $\Gamma_{ij}^{k}(x)$ are functions of point $x$.

*A. Kawaguchi* (Sapporo)

954:

**Laptev, G. F. Invariant normalization of a surface in a space of affine connection.** Dokl. Akad. Nauk SSSR **126** (1959), 490–493. (Russian)

L'A. construit analytiquement une normalisation invariante d'une variété $V_n$ plongée dans un espace à connexion affine; $N \leq n + \frac{1}{2}n(n+1)$. Il utilise sa méthode générale [Trudy Moskov. Mat. Obšč. **2** (1953), 275–382; MR **15**, 254]. *A. Švec* (Prague)

955:

**Stoka, Marius I. Les invariants intégraux d'un groupe de Lie de transformations.** An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat **7** (1958), no. 20, 33–35. (Romanian. Russian and French summaries)

The author proves by direct method a theorem of Deltheil according to which the integral invariants $F$ of a Lie group of transformations satisfy the following system of partial differential equations: $\partial/\partial x^i[\xi_A{}^i(x)F(x)] = 0$.

*G. Soós* (Debrecen)

956:

**Vinberg, È. B. On invariant linear connections.** Dokl. Akad. Nauk SSSR **128** (1959), 653–654. (Russian)

The author states a number of results on homogeneous spaces with an invariant linear connexion [cf. K. Nomizu, Amer. J. Math. **76** (1954), 33–65; MR **15**, 468]. A homogeneous space is understood to consist of a (real or complex) manifold $\mathfrak{B}$ with a transitive group $\mathfrak{G}$ of differentiable transformations; if the isotropy group is completely reducible, the homogeneous space is called completely reducible. Theorem 1: If $\{\mathfrak{B}, \mathfrak{G}\}$ is a completely reducible homogeneous space, $\mathfrak{G}$ is effective and the stabiliser $\mathfrak{H}$ (of some point of $\mathfrak{B}$) has only a finite number of connected components, then the following are equivalent: (i) $\{\mathfrak{B}, \mathfrak{G}\}$ is reductive, (ii) $\{\mathfrak{B}, \mathfrak{G}\}$ admits an invariant linear connexion [cf. Nomizu, loc. cit.], (iii) the Lie algebra of $\mathfrak{H}$ is reductive [cf., e.g., Bourbaki, *Groupes et algèbres de Lie*, ch. I, Actualités Sci. Indust. no. 1285, Hermann, Paris, 1960]. (The only non-trivial part is (iii)⇒(i), which is not proved here.) Theorem 2: A homogeneous space admits an invariant linear connexion, provided that its stabiliser is connected and one-dimensional. Theorem 3: Every locally flat homogeneous space (i.e., with a locally flat invariant linear connexion) with a semisimple group $\mathfrak{G}$ is locally isomorphic to a homogeneous space associated with a transitive linear representation of a simply connected Lie group $\tilde{\mathfrak{G}}$ locally isomorphic to $\mathfrak{G}$. For the case of simple complex Lie groups tables are given showing which representations occur. *P. M. Cohn* (Manchester)

957:

**Sen, D. K. A complex non-analytic manifold and conformal Minkowski space-time.** Canad. J. Phys. **38** (1960), 145–148.

The classical definition of a complex manifold is given and a Lorentz metric on a 1-dimensional complex manifold is expressed in terms of complex coordinates.

*S. Kobayashi* (Vancouver, B.C.)

958:

**Murakami, Shingo. Sur certains espaces fibrés principaux holomorphes admettant des connexions holomorphes.** Osaka Math. J. **11** (1959), 43–62.

From the author's introduction: "Ce mémoire est consacré à l'étude des espaces fibrés principaux holomorphes dont le groupe est abélien connexe et dont la base est un tore complexe. Dans la première partie, on montre que pour ces espaces fibrés l'existence d'une connexion holomorphe est équivalente à l'existence d'un groupe transitif connexe d'automorphismes. On montre de plus que la forme de courbure d'une connexion holomorphe est alors déterminée par l'espace fibré. Dans la seconde partie, on étudie le groupe $P$ des classes d'espaces fibrés dont le groupe est un groupe de Lie abélien connexe, dont la base est un tore complexe, et qui possèdent une connexion holomorphe. On démontre que ce groupe $P$ est canoniquement isomorphe à la somme directe du sous-groupe $P^0$ des classes d'espaces fibrés possédant une connexion holomorphe intégrable et d'un groupe abélien $P^*$ qui s'interprète comme groupe des formes de courbure. On indique enfin la structure de ces deux groupes facteurs; alors que $P^0$ est le quotient d'un groupe de Lie abélien complexe connexe par un sous-groupe de Lie complexe connexe, $P^*$ est un groupe abélien libre de rang fini."

*R. C. Gunning* (Princeton, N.J.)

959:

**Wakakuwa, Hidekiyo. On affinely connected manifolds with homogeneous holonomy group $\mathrm{CL}(n,Q)\otimes T^1$.** Tôhoku Math. J. (2) **11** (1959), 364–375.

Let $A_{4n}$ be a $4n$-dimensional affinely connected manifold whose restricted homogeneous holonomy group is contained in the real representation of $\mathrm{GL}(n,Q)\otimes T^1$, where $\mathrm{GL}(n,Q)$ is the general linear group of the $n$-dimensional vector space over the quaternions. Then $A_{4n}$ admits three almost complex structures $I$, $J$ and $K$ such that $IJ=-JI=K$, $JK=-KJ=I$, $KI=-IK=J$, $\nabla I=-J\otimes\varphi$, $\nabla J=I\otimes\varphi$, $\nabla K=0$, where $\nabla$ denotes the covariant differentiation and $\varphi$ is a covariant vector field. Conversely, if an affinely connected manifold $A_{4n}$ admits almost complex structures $I$, $J$ and $K$ possessing the above properties, then the restricted homogeneous holonomy group of $A_{4n}$ is contained in the real representation of $\mathrm{GL}(n,Q)\otimes T^1$. *S. Kobayashi* (Vancouver, B.C.)

960:

**Willmore, Thomas James. Generalized torsional derivation.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **26** (1959), 649–653.

The torsional derivative introduced by the reviewer [C. R. Acad. Sci. Paris **245** (1957), 1213–1215; MR **19**, 680] for an almost complex structure and later generalised to an almost product structure is now generalised to a structure determined by two tensor fields $h$, $k$ of type 1/1 and a skew-symmetric tensor field $T$ of type 1/2 satisfying the relations

$$hk+kh=-2I,\quad hTk+kTh=2T.$$

The derivative for the almost complex structure determined by $h$ ($h^2=-I$) is recovered when $k=h$ and $T=H$, where $H$ is the torsion tensor defined by $h$. It is also shown that the 4-form given by the skew-symmetric part of the torsional derivative of $H$ can be expressed as a Nijenhuis bracket $[hH, H]$, and that this expression is an invariant of the structure determined by $h$ even when $h$ is not restricted. It is still an open question, however, whether there exists a derivation determined uniquely by an unrestricted $h$. *A. G. Walker* (Liverpool)

961:

**Federer, Herbert. Curvature measures.** Trans. Amer. Math. Soc. **93** (1959), 418–491.

If a compact differentiable manifold, $M$, of dimension $k$ and class $\geq 2$ is imbedded in Euclidean space $E$ of dimension $n>k$, we can form its "tube" by considering all points of $E$ whose distance from $M$ is $\leq r$. The volume of the tube is a polynomial of degree $n$ in $r$ called the Steiner polynomial. The coefficients of this polynomial are (apart from constant factors) Minkowski's "Quermassintegrale", and are expressible as integrals over $M$ of symmetric functions of the principal curvatures, or of scalars derived from the curvature tensor of $M$. Closely related to this theory is the "principal kinematic formula" of integral geometry.

This paper reconsiders the above problems from a more general point of view using measure theory as a principal tool. Previous investigations had developed a theory which established first-order tangential properties of point-sets, and this paper investigates second-order properties such as curvature. The basic notion is that of a set with "positive reach". The reach of a subset $A$ of $E$ is the largest $\varepsilon$ such that if $x\in E$ and the distance $\delta_A(x)$ from $x$ to $A$ is smaller than $\varepsilon$, then $A$ contains a unique point, $\xi_A(x)$, nearest to $x$. The class of such sets includes all convex sets and submanifolds of class $\geq 2$, among others. The Steiner result takes the following form. Assume that the reach of $A$ is positive. For each bounded Borel subset, $Q$, of $E$ and for $0\leq r<\mathrm{reach}(A)$, the $n$-dimensional measure of the generalized tube:

$$E\cap\{x:\delta_A(x)\leq r \text{ and } \xi_A(x)\in Q\}$$

is a polynomial

$$\sum_{i=0}^{n} r^{n-i}\alpha(n-i)\Phi_i(A,Q),$$

where $\alpha(j)$ is the $j$-dimensional measure of a unit spherical ball in $E^j$, and the $\Phi_i$ are generalizations of the Quermassintegrale. In particular, if $A$ is compact, $\Phi_0(A,A)$ is the Euler-Poincaré characteristic of $A$ (Gauss-Bonnet theorem). There is a similar generalization of the kinematic formula of integral geometry. The paper contains a new integral formula connected with a map $f:X\to Y$ where $X$ and $Y$ are Riemannian manifolds of class $\geq 1$ and $\dim X\geq\dim Y$. This generalizes the classical result where $\dim X\leq\dim Y$. *C. B. Allendoerfer* (Seattle, Wash.)

962:

**Coz, Marcel. Une classe de modèles euclidiens pour métriques variationnelles.** Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8°. (2) **31** (1959), no. 4, 42 pp.

The author gives a detailed statement of his result stated in his previous papers [C. R. Acad. Sci. Paris **244** (1957), 1873–1875; **246** (1958), 877–880; MR **19**, 306; **21** #2290] in Chapter I, which deals with a class of models in a euclidean three-dimensional space for the generalized metric given by $ds=f(u, v; du, dv)$, where the function $f$ is positive and homogeneous of degree one in $du, dv$, i.e., for the Finsler metric in a two-dimensional manifold. Chapter II is devoted to the specification of the indicatrix of the metric. In Chapter III degeneration of the model is studied. *A. Kawaguchi* (Sapporo)

## GENERAL TOPOLOGY, POINT SET THEORY

963:

**Gilsdorf, Bernd. ★Über Umgebungssysteme in Moore-schen topologischen Räumen.** Dissertation, Universität Köln. 1958. 17 pp.

A general topological space $(E, \mathfrak{T})$ in the sense of E. H. Moore is a set $E$ and a map $\mathfrak{T}: \mathfrak{P}E \to \mathfrak{P}E$ ($\mathfrak{P}E$ is the power set of $E$). A general neighborhood space $(E, \mathfrak{U})$ is a set $E$ and a map $\mathfrak{U}: E \to \mathfrak{P}\mathfrak{P}E$. Every $(E, \mathfrak{U})$ induces an $(E, \mathfrak{T}_{\mathfrak{U}})$ by means of the following rule: If $M \in \mathfrak{P}E$, then $p \in \mathfrak{T}_{\mathfrak{U}}M \Leftrightarrow$ every $U \in \mathfrak{U}(p)$ meets $M$. [If $M = \emptyset$, $p \in \mathfrak{T}_{\mathfrak{U}}\emptyset \Leftrightarrow \mathfrak{U}(p) = \emptyset$.] The collection $\mathfrak{N}$ of open sets of $(E, \mathfrak{T})$ is defined by the rule: $N \in \mathfrak{N} \Leftrightarrow \mathfrak{T}(E-N) \subseteq (E-N)$. $\mathfrak{N}$ induces a neighborhood topology $\mathfrak{V}$ if we agree that $N \in \mathfrak{V}(p) \Leftrightarrow p \in N$ ($\mathfrak{V}$ is the absolute neighborhood topology of Alexandroff-Hopf). The author proves two theorems: (1) The absolute neighborhood topology induces the original topology (i.e., $\mathfrak{T}=\mathfrak{T}_{\mathfrak{v}}$) $\Leftrightarrow$ Kuratowski's Axioms II and III and a weakened form [$\mathfrak{T}A \cup \mathfrak{T}B \subseteq \mathfrak{T}(A \cup B)$; $A, B \in \mathfrak{P}E$] of Axiom I hold. (2) Call a general neighborhood space satisfying Hausdorff's second axiom a $G$-space, and call a $G$-space satisfying Hausdorff's third axiom an $H$-space. Then for every $G$-space $(E, \mathfrak{U})$ defined on $E$ there is an $H$-space $(E, \mathfrak{X})$ whose topology $\mathfrak{X}$ is weaker than every topology $\mathfrak{X}'$ for which $(E, \mathfrak{X}')$ is an $H$-space and $\mathfrak{X}'$ is stronger than $\mathfrak{U}$. *H. Komm* (Troy, N.Y.)

964:

**Bagley, R. W.; McKnight, J. D., Jr. On $Q$-spaces and collections of closed sets with the countable intersection property.** Quart. J. Math. Oxford Ser. (2) **10** (1959), 233–235.

The authors define an "$I$-space" to be one for which each collection of closed sets with the countable intersection property (c.i.p.) is contained in a maximal such collection. Every countably compact or Lindelöf space is an $I$-space. If $X = Y \cup \bigcup X_n$, where $Y$ is Lindelöf and $X_n$ is closed in $X$ and is an $I$-space ($n = 1, 2, \cdots$), then $X$ is an $I$-space. A normal $I$-space which is also a $Q$-space is Lindelöf (here normality can be slightly weakened but not dispensed with). Hence any uncountable discrete space with non-measurable cardinal is a normal $Q$-space which is not an $I$-space. This raises a problem of constructivity (which seems to the reviewer to have the answer "yes": take for example $X$ = set of all countable ordinals; the collection of sets of the form $\{y \mid y \in X, y > x\}$ ($x \in X$) has the c.i.p. and is not contained in any maximal c.i.p. collection of subsets of $X$, since otherwise one would obtain a non-trivial 2-valued measure on $X$}. Finally the authors observe that, by taking suitable maximal c.i.p. collections of closed subsets of $X$ as the points of a new space, one can imbed each $T_1$ $I$-space $X$ (densely) in a Lindelöf space, and each completely regular $T_1$ space $X$ in a $Q$-space (essentially $v(X)$). *A. H. Stone* (Manchester)

965:

**Ivanova, V. M.; Ivanov, A. A. Contiguity spaces and bicompact extensions of topological spaces.** Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 613–634. (Russian)

Detailed presentation of results announced by the authors in Dokl. Akad. Nauk SSSR **127** (1959), 20–22 [MR **21** #5943]. *J. Isbell* (Seattle, Wash.)

966:

**Gál, I. S. Uniformizable spaces with a unique structure.** Pacific J. Math. **9** (1959), 1053–1060.

The author proves some characterizations of spaces with unique structure, emphasizing the one by uniform approximability of all real continuous functions by functions constant outside a compact set. Some attention is also given to simplifying the earliest published proofs of the main results about these spaces. These expository problems can be better treated in a book; cf. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N.J., 1960. In a note to the reviewer, the author corrects some errors on p. 1054, all arising from the misstatement of E. Hewitt's theorem at the top of the page; see Anais Acad. Brasil. Ci. **21** (1949), 175–179 [MR **11**, 194]. *J. R. Isbell* (Lafayette, Ind.)

967:

**Gorin, E. A. On uniformly topological imbedding of metric spaces in Euclidean and in Hilbert space.** Uspehi Mat. Nauk **14** (1959), no. 5 (89), 129–134. (Russian)

The only new result is a uniform equivalence of $E^n$ into a bounded set in Hilbert space. *J. Isbell* (Seattle, Wash.)

968:

**Restle, Frank. A metric and an ordering on sets.** Psychometrika **24** (1959), 207–220.

The problem is to define a metric on subsets of a finite set. Let $m[S_i]$ be a positive additive set-function and let $D_{ij} = m[(S_i \cup S_j) \cap (\bar{S}_i \cup \bar{S}_j)]$. Then $D_{ij}$ is a metric. Define $b_{ijk}$ ($S_j$ is "between" $S_i$ and $S_k$) if we have $S_i \cap \bar{S}_j \cap S_k$ and $\bar{S}_i \cap S_j \cap \bar{S}_k$ both empty. Theorem: If $b_{ijk}$, then $D_{ij} + D_{jk} = D_{ik}$ (and conversely, if $m[S]$ is 0 only on the empty set). Main theorem: Let $R^*$ be a sequence of sets $R_i$ ($i = 1, \cdots, n$) on which $i \leq j \leq k$ implies $b_{ijk}$. Then $R^*$ must be a "linear array" with the structure: there exist sets $A_i$ and $B_i$ ($i = 1, \cdots, n$) and a set $C$ for which $A_n$, $B_n$ and $C$ are disjoint, and $A_i \subset A_{i+1}$ and $B_i \subset B_{i+1}$ and $R_i = A_i \cup B_{n-i+1} \cup C$. The converse also holds: if $R^*$ has such a decomposition then "betweenness" is transitive; $i \leq j \leq k$ implies $b_{ijk}$. Definitions are given extending these arrays to higher dimensions. In these spaces the

distances between points is the sum of the distances along dimensions (rather than a Pythagorean sum). It is suggested that such arrays and spaces may be useful in psychological scaling methods.

*M. L. Minsky* (Cambridge, Mass.)

969:

**Duda, R. Sur les prolongements ponctiformes des homéomorphies.** Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 132–136.

Generalizing an earlier theorem of de Groot, the author proves that if $A_1$ and $A_2$ are non-dense punctiform sets in compact metric spaces $X_1$ and $X_2$, respectively, and if neither $A_1$ nor $A_2$ separates $X_1$ or $X_2$ locally at any point, then any homeomorphism of $X_1 - A_1$ onto $X_2 - A_2$ may be extended (uniquely) to a homeomorphism of $X_1$ onto $X_2$. If $X_1$ and $X_2$ are compact manifolds, this holds for arbitrary punctiform sets $A_1$ and $A_2$. An extension of the theorem is made to certain types of non-compact manifolds.

*G. T. Whyburn* (Charlottesville, Va.)

970:

**Lelek, A. Ensembles $\sigma$-connexes et le théorème de Gehman.** Fund. Math. **47** (1959), 265–276.

A set $X$ is said to be $\sigma$-connected provided it admits no decomposition into a countable infinity of disjoint non-empty closed sets. A study is made of this basic property; and various related theorems, examples and unsolved problems are given. In particular, it is shown that (1) each $\sigma$-connected set $X$ has only a finite number of components and each of these is $\sigma$-connected and (2) every connected subset of a plane hereditarily locally connected continuum is $\sigma$-connected.

*G. T. Whyburn* (Charlottesville, Va.)

971:

**Titus, Charles J. Sufficient conditions that a mapping be open.** Proc. Amer. Math. Soc. **10** (1959), 970–973.

The author considers the class $F$ of mappings $f$ of $M$ into $N$, where $M$ and $N$ are oriented connected $n$-manifolds, satisfying the conditions: (i) $f$ is sense-preserving at all points of $M$ except points of the set $Z$ in $M$ where it is not locally topological; (ii) $f(Z)$ contains no open sets; (iii) $f$ is constant on every component of the interior of $Z$. It is shown that "any non-trivial mapping in $F$ is quasi-open and that certain convenient differential inequalities imply that a mapping is in $F$". Specifically, if $M$ and $N$ are $C'$ and $f$ is a $C'$ mapping such that the linear transformation $df$ never reverses sense and has rank zero at all interior points of the set where it is singular, then $f$ is in $F$.

*G. T. Whyburn* (Charlottesville, Va.)

972:

**Jurchescu, Martin. On a theorem of Stoilow.** Math. Ann. **138** (1959), 332–334.

S. Stoïlow has shown that a 2-manifold $X$ has a countable topology if there exists an interior mapping of $X$ into the sphere $S^2$ [*Leçons sur les principes topologiques de la théorie des fonctions analytiques*, Gauthier-Villars, Paris, 1956; MR **18**, 568]. Using the basic idea of Stoïlow's proof, the author gives the following generalization: A connected, locally connected, locally compact, and locally countable Hausdorff space has a countable topology if it admits a 0-dimensional mapping into a Hausdorff space with a countable topology. As a corollary, a manifold or a connected complex space has a countable topology if it can be 0-dimensionally mapped into $C^\omega$. From this the author derives a simplified proof of Grauert's theorem [Math. Ann. **129** (1955), 233–259; MR **17**, 80]: Any connected $K$-complete space has a countable topology.

*L. Sario* (Los Angeles, Calif.)

973:

**Tanaka, Tadashi. On certain continuous and equicontinuous collections of compact continua.** J. Sci. Hiroshima Univ. Ser. A **22**, 191–203 (1958).

The theorems of this paper are special cases of previously known theorems; e.g., Theorem 2 is the special case of Theorem 7 of Dyer and Hamstrom, Fund. Math. **45** (1958), 103–118 [MR **19**, 1187], in which $K$ is a 1-cell and $Y$ is an $i$-cell, $i = 0$, 1 or 2; and Theorem 3 is the special case of the first part of Theorem 8 of Dyer, Ann. of Math. (2) **67** (1958), 119–149 [MR **19**, 1071] in which the spaces involved are metric and $n = k = 0$.

*E. Dyer* (Chicago, Ill.)

974:

**Gary, John. Higher dimensional cyclic elements.** Pacific J. Math. **9** (1959), 1061–1070.

Important extensions are given of earlier results of the reviewer, Puckett and Simon concerning the higher order cyclic element decomposition in the setting of a compact Hausdorff space $M$. Structure theorems concerned with the relation between homology and cohomology groups of $M$ and those of the cyclic elements of appropriate dimensionality are obtained. These are formulated in terms of the natural inclusion mappings and constitute interesting extensions of cases treated formerly by other methods. The action of the cyclic elements under higher order monotone mappings is also studied. It is shown by an example that the basic result concerning ordinary cyclic elements, asserting that given a monotone mapping of $M$ onto $N$, each cyclic element of $N$ is covered by the image of some cyclic element of $M$, does not extend in this form. However, the author does obtain a somewhat analogous theorem relating the carrier of a cocycle in $N$ and those of its counter image in $M$ when the basic mapping is of the corresponding order of monotoneity.

*G. T. Whyburn* (Charlottesville, Va.)

975:

**Suvorov, G. D. A theorem on the sequences of topological mappings of such regions as belong to compacta.** Dokl. Akad. Nauk SSSR **129** (1959), 744–746. (Russian)

Extension of the classical notion (Carathéodory) of the kernel of a sequence of regions to the case of a sequence of regions lying in a compact metric space. The classical theorem (of Carathéodory) concerning a sequence of regions converging to its kernel and conformal maps of the regions is extended to the new setting with topological maps of a certain class taking over the rôle of the conformal maps.

*M. H. Heins* (Urbana, Ill.)

976:

**Connell, E. H. Properties of fixed point spaces.** Proc. Amer. Math. Soc. **10** (1959), 974–979.

The author studies the relation between the fixed point property (f.p.p.) and compactness. He shows that the

familiar space of the unit interval with open sets defined as the usual open sets with the possible deletion of a countable number of points has the f.p.p. In a metric space with the f.p.p., however, a weak compactness condition (but not compactness) is necessary. The condition is that every locally finite chain of (Jordan) arcs be finite. This result is used to show that the f.p.p. is not invariant under the cross product. Another example (due to Strother) shows that the f.p.p. is not invariant under closure. *H. Komm* (Troy, N.Y.)

## ALGEBRAIC TOPOLOGY

See also 974, B1064.

977:

**Greever, John. Stationary points for finite transformation groups.** Duke Math. J. **27** (1960), 163–170.

Let $(G, X)$ be a transformation group in which $G$ is finite and $X$ is a finite-dimensional compact Hausdorff space. Various sets of sufficient conditions are given for the existence of at least one fixed point (i.e., point fixed under all the transformations of the group). Each set contains (1) the assumption that $X$ is homologically or cohomologically trivial over specified coefficients; (2) the condition that $G$ admit a normal chain of specified type; (3) the hypothesis that the integral cohomology groups of the set $F_H$ of fixed points of the subgroup $H$ be finitely generated for each $H$ appearing in the normal chain in (2). The quotient groups which appear in (2) are of prime power orders and at most three distinct primes occur. Suppose in particular that the integral cohomology groups of $F_H$ are finitely generated for every subgroup $H$ of $G$ and that those of $X$ are trivial. The theorems above then show that $F_G$ is non-empty when $G$ is abelian and $[G:1] \leqq 210$, also when $G$ is non-abelian and $[G:1]$ is less than 60 and unequal to 36. *P. A. Smith* (New York)

978:

**Kodama, Yukihiro. A relation between two realizations of complete semi-simplicial complexes.** Proc. Japan Acad. **33** (1957), 536–540.

Let $K$ be a c.s.s. complex. Denote by $\bar{K}$ the topological space which is the (discrete) union of copies of the standard $q$-simplex $\Delta_q$, one for each $q$-simplex of $K$ ($q = 0, 1, 2, \cdots$). Giever [Ann. of Math. (2) **51** (1950), 178–191; MR **11**, 379] and Hu [Pacific J. Math. **1** (1951), 583–602; MR **13**, 676] have discussed the identification space $P(K)$ obtained from $\bar{K}$ by the equivalence relation corresponding to face maps in $K$. Milnor [Ann. of Math. (2) **65** (1957), 357–362; MR **18**, 815] has discussed the space $|K|$ obtained by using both face and degeneracy maps. There is then a natural projection $f: P(K) \to |K|$. It is shown here that $f$ is a homotopy equivalence. The argument is similar to that in the author's paper Fund. Math. **45** (1958), 217–227 [MR **20** #6096]. *J. A. Zilber* (Providence, R.I.)

979:

**Brown, Edgar H., Jr.; Copeland, Arthur H., Jr. An homology analogue of Postnikov systems.** Michigan Math. J. **6** (1959), 313–330.

This paper contains a study of the process of homology decomposition of a 1-connected polyhedron [Moore, *Séminaire H. Cartan 1954/55*, Paris, 1955; MR **19**, 438; Eckmann and Hilton, C. R. Acad. Sci. Paris **248** (1959), 2054–2056; Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 372–375; MR **21** #2977, #7503]. Of particular interest is the authors' demonstrations by means of a simple example that a homotopy type may be decomposed in two essentially different ways. In this respect the homology decomposition differs from, and is inferior to, the Postnikov (homotopy) decomposition to which it stands in conceptual duality.

Let $\mathscr{C}$ be the class of 1-connected polyhedra with 2 non-vanishing homology groups. A homotopy classification of the members of $\mathscr{C}$ is given and a number of the groups $[X, Y]$ of based homotopy classes of based maps $X \to Y$, where $X$, $Y$ belong to $\mathscr{C}$, are computed, extending the results of Barratt [Proc. London Math. Soc. (3) **5** (1955), 285–329; MR **17**, 395]. *P. J. Hilton* (Birmingham)

980:

**Stasheff, James. On the space-of-loops isomorphism.** Proc. Amer. Math. Soc. **10** (1959), 987–993.

The main result of this paper is that if $X$ is an $(n-1)$-connected CW-complex and $\pi_i(Y) = 0$ for $i > 2n-2$, then the set $\Pi(X, Y)$ stands in (1, 1) correspondence with the set $\Pi(\Omega X, \Omega Y)$ and thereby acquires a natural abelian group structure.

If $Z$ is the $(n-1)$-connected fibre space over $Y$ then $\Pi(X, Y) = \Pi(X, Z)$. Moreover $Z$ is equivalent to a loop-space (this is the author's Lemma 1 but may be proved just by observing that $Z \to \Omega\Sigma Z$ induces homotopy isomorphisms). This gives $\Pi(X, Y)$ a natural group structure. It may then be proved (by homology arguments) that $\Sigma\Omega X \to X$ induces an isomorphism $\Pi(X, Y) \cong \Pi(\Sigma\Omega X, Y)$ so that $\Pi(X, Y) \cong \Pi(\Omega X, \Omega Y) \cong \Pi(\Omega X, \Omega Z)$ and the group structure is abelian since $\Omega Z$ is equivalent to a double loop space. The author points out that the condition of validity of the main theorem is in a sense dual to the condition for the existence of generalized cohomotopy groups. *P. J. Hilton* (Birmingham)

981:

**Thomas, Emery. The suspension of the generalized Pontrjagin cohomology operations.** Pacific J. Math. **9** (1959), 897–911.

Soit $A$ un anneau gradué à puissances divisées, tel que chaque composante homogène $A_k$ soit cyclique d'ordre infini ou une puissance d'un nombre premier. E. Thomas [Mem. Amer. Math. Soc. no. 27 (1957); MR **20** #5474] a défini, pour chaque entier $t \geqq 0$, des "opérations de Pontrjagin" $\mathfrak{P}_t: H^{2n}(X; A_{2k}) \to H^{2n+t}(X; A_{2tk})$; $\mathfrak{P}_2$ est le carré de Pontrjagin proprement dit. L'auteur définit ici, en toute généralité, des "opérations de Postnikov" $\mathfrak{p}: H^q(X; A_{2k}) \to H^{2q+1}(X; A_{4k})$ et démontre le théorème suivant: la suspension $S(\mathfrak{P}_t): H^{2n-1}(X; A_{2k}) \to H^{2n+t-1}(X; A_{2tk})$ est nulle si $t \geqq 3$, et $S(\mathfrak{P}_2) = \mathfrak{p}$. De plus, $S(\mathfrak{p}) = 0$. On donne en outre une liste des propriétés de $\mathfrak{p}$, et notamment celles-ci: $\mathfrak{p} \circ \mathfrak{p} = 0$; si $A_{2k}$ est d'ordre $2^t$ ($i \geqq 2$) et si $u \in H^q(X; A_{2k})$ est tel que $2u = 0$, alors $\mathfrak{p}(u) = 0$. *H. Cartan* (Paris)

982:

**Koszul, J. L. Espaces fibrés associés et pré-associés.** Nagoya Math. J. **15** (1959), 155–169.

Dans une première partie l'auteur complète et généralise les résultats d'un travail antérieur [Trans. Amer. Math. Soc. **89** (1958), 256–266; MR **20** #6099]. Pour tout couple d'espaces topologiques $X$, $Y$, soit Hom $(X, Y)$ le groupe abélien libre ayant pour base l'ensemble des applications continues $X \to Y$; la composition

$$\text{Hom}\,(Y, Z) \otimes \text{Hom}\,(X, Y) \to \text{Hom}\,(X, Z)$$

est définie par linéarité à partir de la composition des applications continues. Un "complexe topologique" $X_*$ est défini par la donnée d'une suite d'espaces topologiques $X_q$ ($q$ entier) et, pour chaque $q$, d'un $d_q \in \text{Hom}\,(X_q, X_{q-1})$, avec $d_{q-1}d_q = 0$. Soit $S_p(X_q)$ le groupe des chaînes singulières de dimension $p$ de $X_q$; la somme directe $S_*(X_*) = \sum_{p,q} S_p(X_q)$ est munie d'une structure de complexe double, avec deux opérateurs $\partial' : S_p(X_q) \to S_{p-1}(X_q)$, $\partial'' : S_p(X_q) \to S_p(X_{q-1})$ tels que $\partial'\partial' = 0$, $\partial''\partial'' = 0$, $\partial'\partial'' + \partial''\partial' = 0$. Pour tout anneau commutatif $A$, on note $H_*(X_*; A)$ [resp. $H^*(X_*; A)$] l'homologie [resp. la cohomologie] à coefficients dans $A$ du complexe double $S_*(X_*)$ muni de la différentielle $\partial' + \partial''$. A chaque espace $Y$ on associe le complexe topologique $X_* = R(Y)$ tel que $X_q = Y^{q+1}$ pour $q \geq 0$, $X_q = \emptyset$ pour $q < 0$, avec une différentielle évidente. On lui associe aussi le complexe topologique, encore noté $Y$, défini par $Y_0 = Y$, $Y_q = \emptyset$ pour $q \neq 0$; alors $H_*(Y; A)$ s'identifie à l'homologie singulière de l'espace $Y$, et de même pour la cohomologie. On démontre que

$$H_p(R(Y); A) = 0 \quad \text{pour } p \neq 0, \quad H_0(R(Y); A) = A.$$

Si un groupe topologique $G$ opère (à droite) continûment dans $Y$, il opère dans $R(Y)$, d'où un complexe topologique quotient $R(Y)/G$; on montre (théorème 1) que si deux applications continues $Y \to Y'$ commutent aux opérations de $G$, les homomorphismes $S_*(R(Y)/G) \to S_*(R(Y')/G)$ qu'elles définissent sont homologues. Ceci permet de définir, pour tout fibré principal $Y$ de groupe $G$, un isomorphisme canonique $H^*(R(Y)/G; A) \approx H^*(R(G)/G; A)$ (th. 2), et $H^*(R(Y)/G; A)$ s'identifie à la cohomologie d'un espace "classifiant" du groupe $G$. Alors l'injection canonique $Y/G \to R(Y)/G$ définit un homomorphisme

$$H^*(R(Y)/G; A) \to H^*(Y/G; A)$$

qui s'identifie à l'homomorphisme "caractéristique" du fibré principal $Y$ (th. 3). Pour chaque entier $n \geq 0$, soit $K^n(Y)$ le complexe topologique $X_*$ tel que $X_q = R_q(Y)/G$ pour $q \leq n$, $X_q = \emptyset$ pour $q > n$; on a donc $K^0(Y) = Y/G$; l'injection $K^0(Y) \to K^n(Y)$ définit $H^*(K^n(Y); A) \to H^*(Y/G; A)$. Soit $Q_n$ l'image de cet homomorphisme; les $Q_n$ définissent une filtration décroissante de $H^*(Y/G; A)$; $Q_\infty$ est le module des classes caractéristiques du fibré principal $Y$, et $Q_1$ est le noyau de $H^*(Y/G; A) \to H^*(Y; A)$.

Dans une deuxième partie, on considère, lorsque $X = Y/\Gamma$ est la base d'un fibré principal $Y$ de groupe $\Gamma$, l'ensemble $H^1(X, \mathfrak{G})$ des classes de fibrés principaux de base $X$ et de groupe $G$. On définit une filtration $Q_2 \subset Q_1 \subset H^1(X, \mathfrak{G})$; les fibrés principaux $P$ qui correspondent aux éléments de $Q_2$ sont ceux qui sont "associés" à $Y$ pour un homomorphisme (continu) convenable $\Gamma \to G$; les fibrés $P$ qui correspondent aux éléments de $Q_1$ sont dits "pré-associés": ce sont ceux tels que les deux fibrés de base $Y^2/\Gamma$, images réciproques de $P$ par les deux applications $Y^2/\Gamma \to Y/\Gamma$ déduites des deux projections $Y^2 \to Y$, soient isomorphes. Tout fibré pré-associé de base $X$ est trivialisé par la projection $Y \to X$ du fibré $Y$. L'auteur fait une étude détaillée de $Q_1$ et $Q_2$, ainsi que des "facteurs" $Y \times \Gamma \to G$. Il étudie le cas où $Y$ est un fibré principal différentiable, $P$ étant pré-associé à $Y$; lorsque $\Gamma$ est discret, il étudie les "formes de connexion" sur $P$ et les groupes d'holonomie correspondants. Le cas des fibrés holomorphes est aussi envisagé.

*H. Cartan* (Paris)

983:

**Wu, Wen-tsün. On certain invariants of cell-bundles.** Sci. Record (N.S.) **3** (1959), 137–142.

For an $n$-cell bundle over a Hausdorff space $B$ (without structural group), the author defines certain invariants $Q_p{}^i \in H^i(B; Z_p)$ by means of Smith classes and the Leray-Hirsch theorem, which turn out to be characteristic classes in case the bundle has the linear group as its structural group. More precisely, $Q_2{}^i$ is the $i$th Stiefel-Whitney class and $Q_p{}^i$ for $p > 2$ is a function of the Pontrjagin classes mod $p$ of the bundle in case the linear structure group is given. The author applies this theory to show that for differentiable structures on a topological manifold, the Stiefel-Whitney classes and certain combinations of the Pontrjagin classes mod $p$ of the manifold do not depend on the differentiable structure. Thus in this case, the $Q_p{}^i$ are defined intrinsically and are topological invariants, recovering a theorem of Thom.

*S. Smale* (Berkeley, Calif.)

984:

**Kinoshita, Shin'ichi. Alexander polynomials as isotopy invariants. II.** Osaka Math. J. **11** (1959), 91–94.

Let $C$ be a compact 3-dimensional manifold whose boundary is the union of $\nu$ orientable surfaces $M_1, \cdots, M_\nu$. Then the nullity of the Alexander matrix $\mathbf{A}$ of $C$ is $\geq d = 1 + \sum_{k=1}^{\nu} (h_k - 1)$, where $h_k$ is the genus of $M_k$. Let $F(C)$ denote the fundamental group of $C$. The set $\Phi$ of all homomorphisms $\phi$ of $F(C)/[F(C), F(C)]$ into the infinite cyclic group $Z$ is a free abelian group. If $\phi \in \Phi$, define $\Delta_\phi$ to be the greatest common divisor, in the integral group ring of $Z$, of the minor determinants of $\mathbf{A}^\phi$ of column defect $d$. Let $\phi_1, \cdots, \phi_p$ be a basis for $\Phi$, and denote by $\#(C)$ the number of non-constant polynomials in the set $\Delta_{\phi_1}, \cdots, \Delta_{\phi_p}$; $\#(C)$ is called by the author the number of arbitrary constants of the Alexander polynomial of $C$, and he proves that, if $M_1, \cdots, M_\mu$ are disjoint tame closed surfaces in euclidean 3-space $S$ and $C_0, C_1, \cdots, C_\mu$ are the closures of the components of $S - (M_1 \cup \cdots \cup M_\mu)$, then $\sum_{i=0}^{\mu} \#(C_i) \leq 2\sum_{i=1}^{\mu} h_i - 1$. From this follows the interesting corollary: If $C$ and $E$ are compact 3-manifolds such that their boundaries $M$ and $N$ are connected surfaces of the same genus $h$, and if $\#(C) = \#(E) = h$, then $C$ and $E$ do not make a 3-sphere by any identification of $M$ and $N$.

*R. H. Fox* (Princeton, N.J.)

985:

**Cairns, S. S. Circumscribed cubes in euclidean $n$-space.** Bull. Amer. Math. Soc. **65** (1959), 327–328.

The author investigates a problem of finding a circumscribing cube around a given bounded set in an $n$-dimensional euclidean space $E^n$. The approach is similar to one taken by A. Heller [Ann. of Math. (2) **55** (1952), 223–231; MR **14**, 1110]. Namely, the author reduces the problem to the one of finding an inverse image, denoted

by $K$, of a map from $R_{n-1}$, the rotation group of axes of $E^n$, into a closed $(n-1)$-cell. This set $K$ virtually represents the set of circumscribing cubes. The author concludes that since $K$ is known not to be empty it has the dimension at least $(n-1)(n-2)/2$. The author also gives a conjecture that $K$ contains a subset homeomorphic with $R_{n-2}$. *H. Yamabe* (Evanston, Ill.)

986:

**Dyer, E. On the dimension of products.** Fund. Math. **47** (1959), 141–160.

For $p$ a prime let $R_p$ denote the additive group of those fractions which in lowest form have denominators which are not divisible by $p$. Let $D(X; G)$ denote the cohomological dimension of $X$ with coefficients in $G$. Say that a space $X$ is cohomologically locally connected in all dimensions through $n$ (written clc$^n$) over $G$ if for each $x$ in $X$ and each closed neighborhood $U$ of $X$, there is a closed neighborhood $V$ of $x$ such that $V \subset U$ and the inclusion homomorphism from $\bar{H}^i(U; G)$ to $\bar{H}^i(V; G)$ is trivial for all $i \leq n$. The author shows that if dim $(X \times Y) =$ dim $X +$ dim $Y$ then there is a prime $p$ such that $D(X; R_p) =$ dim $X$ and $D(Y; R_p) =$ dim $Y$ (where dim denotes the covering dimension). Conversely he shows that if $X$ and $Y$ are clc$^n$ over $Z$ for all $n$ and if there is a prime $p$ with $D(X; R_p) =$ dim $X$ and $D(Y; R_p) =$ dim $Y$, then dim $(X \times Y) =$ dim $X +$ dim $Y$.

Of additional interest are some algebraic lemmas leading to somewhat streamlined proofs of some theorems of Bokšteĭn [Trudy Moskov. Mat. Obšč. **5** (1956), 3–80; **6** (1957), 3–133; MR **18**, 813; **19**, 875].

*Haskell Cohen* (Baton Rouge, La.)

## DIFFERENTIAL TOPOLOGY

See also 946, 951, 952, 956, 957, 958, 959, 960.

987:

**Bott, Raoul. The stable homotopy of the classical groups.** Ann. of Math. (2) **70** (1959), 313–337.

This is a detailed account of the proof of the important results of the author on the stable homotopy groups of the classical Lie groups as announced in an earlier note [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 933–935; MR **21** #1588]. The main steps in the treatment of the various groups have been unified in a single general proposition (Theorem I) which contains the periodicity theorems as special cases. This theorem and its proof are a striking illustration of the fact that geometrical methods in homotopy theory still have some interest.

Theorem I is as follows. Let $M$ be a closed compact connected Riemannian manifold of class $C^\infty$. Let $\nu = (P, Q, h)$ consist of a pair of points $P, Q$ on $M$ and a homotopy class $h$ of paths from $P$ to $Q$. Let $\Omega_\nu M$ be the set of piecewise differentiable curves from $P$ to $Q$ on $M$, in the homotopy class $h$, parametrized from 0 to 1 proportionally to arc length. $\Omega_\nu M$ is given the topology induced by the metric

$$\rho(c, c') = \max \rho_0(c(t), c'(t)) + |J(c) - J(c')|,$$

where $\rho_0$ is the distance on $M$ and $J(c)$ is the length of $c$. If $s \in \Omega_\nu M$ is a geodesic segment and $t \in I$, denote by $\lambda(s(t))$ the dimension of the vector space of Jacobi fields along $s$ vanishing at $P = s(0)$ and $s(t)$. (The definition of a Jacobi field is recalled on page 319.) Define the index of a geodesic segment $s \in \Omega_\nu M$ to be the finite number $\sum_{0<t<1} \lambda(s(t))$. Let $|\nu|$ denote the minimum of the positive integers which occur as the index of a geodesic segment in $\Omega_\nu M$, and let $M^\nu$ be the subspace of $\Omega_\nu M$ consisting of the geodesic segments of minimal length. Theorem I: If $M$ is a symmetric space (in the sense of E. Cartan), then $M^\nu$ is also a symmetric space (in particular a Riemannian manifold), and the inclusion $i: M^\nu \to \Omega_\nu M$ induces isomorphisms $i_*: \pi_k(M^\nu) \to \pi_k(\Omega_\nu M) \approx \pi_{k+1}(M)$ for $0 < k < |\nu| - 1$. In dimension $k = |\nu| - 1$, $i_*$ is an epimorphism.

A procedure to compute $|\nu|$ in practice is the subject of § 6.

The proof of Theorem I is essentially in two steps. Step I: In § 3, a homotopy formulation of Morse theory is given following an idea inaugurated by R. Thom [C. R. Acad. Sci. Paris **228** (1949), 973–975; MR **10**, 558]. Thom's argument is made more specific using J. H. C. Whitehead's notion of a CW-complex and is extended so as to include a more general type of critical sets, i.e., nondegenerate critical manifolds, a notion due to the author [Ann. of Math. (2) **60** (1954), 248–261; MR **16**, 276]. Step II: Previous work of the author, and joint work with H. Samelson is then used to show that the Morse theory in its new formulation applies to the symmetric spaces and yields the result.

The paper ends with explicit computations of $|\nu|$ for the symmetric spaces arising from the classical Lie groups (§ 7 and § 8).

Although results from Morse theory and previous work of the author are used, the proofs can be followed if these results are taken for granted. Moreover, precise references are given whenever the proofs are not self-contained and the reconstitution of the full details is a routine matter.

*M. A. Kervaire* (New York)

988:

**Borel, A.; Hirzebruch, F. Characteristic classes and homogeneous spaces. II.** Amer. J. Math. **81** (1959), 315–382.

This second part of the authors' paper on characteristic classes [Part I: same J. **80** (1958), 458–538; MR **21** #1586] contains the main applications of their theory. These applications stem essentially from the fusion of the methods developed by Hirzebruch for his proof of the Riemann-Roch theorem, and the special knowledge of the characteristic classes of homogeneous bundles, due in so large part to A. Borel. The results, whose proofs are here presented for the first time, have already had a profound effect on the development of the whole subject during the last few years. They have served as a foundation or guidepost to many and especially to this reviewer. The several anachronistic aspects of the paper are ample evidence for this influence. They are also a tribute to the generosity with which the authors discussed their results.

One of the most influential of these results has been the divisibility theorem of section 25. It might therefore be worth while to freely transcribe here the main ideas of this proof and to report also on those steps which are to be found in Hirzebruch's work and which the authors assume, a trifle too abruptly it seemed to this reviewer, as their starting point.

If $X$ is a space, $F_R(X)$ [$F_C(X)$] shall denote the set of

$O(n)$- [$U(n)$-] bundles over $X$ ($n=1, 2, \cdots$). The usual inclusion $O(n)\times O(m)\subset O(n+m)$ defines an operation in $F_R(X)$ which is denoted by $+$ and referred to as the Whitney sum. A function $Q: F_R(X)\to H^{**}(X)$ for which $Q(\xi+\eta)=Q(\xi)\cdot Q(\eta)$ is called a multiplicative function on $F_R(X)$. (Throughout, $H^{**}(X)$ denotes the direct product of the cohomology groups $H^i(X; R)$ with coefficients in the real numbers.) In a quite analogous manner one speaks of multiplicative functions on $F_C(X)$.

The basic multiplicative function in the whole theory of characteristic classes is the Chern class $c: F_C(X)\to H^{**}(X)$. (In an appendix the authors perform the public service of comparing seven prevalent definitions of this function as to sign. They also discuss the Pontryagin class from different points of view in a second appendix.) All other natural multiplicative functions from $F_C(X)$ to $H^{**}(X)$ can be described in terms of the components of the Chern class (as follows from the fact that the universal ones generate the whole cohomology rings of the classifying space.) Hirzebruch pointed out that multiplicative functions can be generated in the following manner.

Let $Q(x)=1+a_1x+a_2x^2\cdots$ be a formal power series with real coefficients. If $\eta\in F_C(X)$ is a $U(n)$-bundle, rewrite $Q(x_1)\cdots Q(x_n)$ as a power series in the elementary symmetric functions $c_i=\sum x_1\cdots x_i$, and then set $c_i$ equal to $c_i(\eta)$—the $2i$-dimensional component of $c(\eta)$—thus obtaining a class $Q(\eta)$ in $H^{**}(X)$. He also established that if $T$ is the multiplicative function determined by

$$T(x)=\frac{x}{1-\varepsilon^{-x}}=\frac{\frac{1}{2}xe^{x/2}}{\sinh\frac{1}{2}x}$$

then $T(t_c)$ has an integral value on the orientation class of any projective nonsingular variety over the complex numbers, with complex tangent bundle $t_c$.

Hirzebruch of course showed more; namely, that this integer—to be written $T(t_c)[X]$—was equal to the arithmetic genus of the variety. However in this paper it is simply the integrality of $T(t_c)$ on $X$ which is a motivating factor, the aim being to prove an integrality theorem of this type for differentiable manifolds in general. Let $\hat{A}(x)=\frac{1}{2}x/\sinh\frac{1}{2}x$, so that $T(x)=e^{x/2}\hat{A}(x)$. The series $\hat{A}(x)$ clearly has the property that $\hat{A}(x)=\hat{A}(-x)$. From this, it follows quite easily that the value, $\hat{A}(\eta)$, of this multiplicative function depends only on the $O(2n)$-bundle determined by the $U(n)$-bundle $\eta$ according to the inclusion $U(n)\subset O(2n)$. To see this, let $\varepsilon: F_C(X)\to F_R(X)$ be the operation just described and let $\rho: F_R(X)\to F_C(X)$ be induced by the inclusions $O(2n)\subset U(2n)$. Also let $\eta\to\bar{\eta}$ be the operation induced by complex conjugation in $U(n)$. From the elementary fact that if $Q(x)=1+ax\cdots$ determines the function $Q$, then $Q(\bar{\eta})=\bar{Q}(x)$, where $\bar{Q}(x)=Q(-x)$, it follows that $\hat{A}(\eta)=\hat{A}(\bar{\eta})$. Now let $\hat{A}^{1/2}(x)=1+b_1x^2\cdots$ be the formal square root of $\hat{A}(x)$; given any $\xi\in F_R(X)$ define $\hat{A}(\xi)$ as $\hat{A}^{1/2}(\rho(\xi))$. This defines a multiplicative function $\hat{A}$ on $F_R(X)$ which agrees with the definition of $\hat{A}$ on $F_C(X)$; that is, $\hat{A}(\eta)=\hat{A}(\varepsilon\eta)$ for any $\eta\in F_C(X)$. Indeed, $\hat{A}(\varepsilon\eta)=\hat{A}^{1/2}(\rho\varepsilon\eta)=\hat{A}^{1/2}(\bar{\eta})\cdot\hat{A}^{1/2}(\eta)=\hat{A}(\eta)$. (In general this procedure yields a multiplicative function $Q: F_R(X)\to H^{**}(X)$ whenever the power series $Q(x)$ has the property $Q(x)=Q(-x)$; one defines $Q(\xi)$ for $\xi\in F_R(X)$ as $Q^{1/2}(\rho\xi)$.)

The multiplicative function $\hat{A}$ on $F_R(X)$ is therefore a reasonable candidate for an integrality theorem valid on differentiable manifolds, and the authors in fact prove such a theorem, except that in the real case integrality can be established only for the odd primes. To express their results precisely it is convenient to apply the term "exc 2" to real numbers (or real cohomology classes), whenever these quantities become integral after multiplication by a suitable power of 2. One more convention has to be recorded before the central integrality theorem of this paper can be stated. In the formalism already used to describe the multiplicative functions, one sets $\mathrm{ch}(\eta)$ equal to $\sum_1^n e^{x_i}$ for any $U(n)$ bundle $\eta$. This "character" is an additive function from $F_C(X)$ to $H^{**}(X)$. With these conventions settled, the authors' theorem (25.5) states the following:

> Let $X$ be an oriented compact manifold, with tangent bundle $t\in F_R(X)$. Let also $\eta\in F_C(X)$ and let $d$ be an integral class exc 2. Then the value of $\mathrm{ch}(\eta)e^d\hat{A}(t)$ on the orientation cycle of $X$ is an integer exc 2.

The consequences of this theorem are highly nontrivial even for simple manifolds. For instance, if $X$ is the $2n$-sphere, then $t+(\text{trivial bundle})=\text{trivial bundle}$, whence $\hat{A}(t)=1$, so that by their theorem ch $\eta$ must be integral, exc 2, on the orientation class of $X$; or, quite equivalently, the $n$th Chern class of a $U(n)$-bundle over $X=S_{2n}$ is divisible by $(n-1)!$ exc 2.

The prime antecedent of this divisibility theorem is undoubtedly the Hirzebruch index formula, and I will try to outline its evolution from that point.

Let $L(x)=x/\tanh x$. Clearly $L(x)=L(-x)$, so that $L$ defines a multiplicative function on $F_R(X)$ by the procedure already described. Hirzebruch shows that if $X$ is a compact oriented differentiable manifold, with tangent bundle $t\in F_R(X)$, then the value of $L(t)$ on the orientation class of $X$ is equal to the index of $X$, that is, $L(t)[X]$ equals the difference of the number of positive and negative eigenvalues of the quadratic form $\varphi(u, v)=(u\cup v)[X]$, $u, v\in H^{**}(X)$. (The proof of this theorem is magical enough: From the cobordism theory of Thom it is first shown that a power series $L(x)$ with this property must exist. Then it is seen that $x/\tanh x$ is the only power series giving the correct answer on the complex projective spaces.)

In any case then, $L(t)[X]$ is an integer, if $t$ is the tangent bundle of $X$. Next Hirzebruch shows that if $v\in H^2(X; Z)$ then $\tanh vL(t)$ is again integral on the orientation class of $X$; indeed he shows that this number is the index of the submanifold dual to $v$ under Poincaré duality. The proof is instructive in that it shows how the nature of the power series $L(x)$ enters the picture. Let $\eta$ be the $U(1)$-bundle determined by $v$, let $E$ be the associated line bundle, and let $s$ be a generic section of $E$. The zeros of $s$ then determine a submanifold $Y\subset X$ of codimension 2, dual to $v$. Let $t_Y$ be the tangent bundle along $Y$. Then $t\,|\,Y=t_Y+\varepsilon\eta$. Hence if $i: Y\to X$ is the injection, then

$$i^*\left\{\frac{\tanh v}{v}L(t)\right\}=L(t_Y)L(\varepsilon\eta)i^*\frac{\tanh v}{v}.$$

However, $c_1(\eta)=v$, whence $L(\varepsilon\eta)=L(\eta)=v/\tanh v$, so that the right-hand side is actually $L(t_Y)$. Hence the index of $Y$ is given by

$$i^*\left\{\frac{\tanh d}{d}L(t)\right\}\cdot[Y]=\tanh vL(t)[X]$$

—the last step being a consequence of Poincaré duality.

Q.E.D. Clearly this procedure can be iterated so that, if $v_1, \cdots, v_k$ are elements of $H^2(X; Z)$, then

$$\prod_1^k (\tanh v_i) L(t)[X]$$

is integral. (These are the virtual indexes of Hirzebruch.) At this point one has a vast array of a priori nontrivial integrality conditions at one's disposal; however, they are interesting only if $H^2(X; Z)$ is richly endowed. Accordingly Hirzebruch first investigates the almost complex split manifolds. Let then $X$ be such an object, so that, by definition, there is a $U(n)$-bundle $\eta$ over $X$, $\eta = \eta_1 + \eta_2 + \cdots + \eta_n$, which is the direct sum of $n$ $U(1)$-bundles $\eta_i$, and such that $\varepsilon\eta = t$, where $t$, as throughout, is the tangent bundle of $X$. Let $x_i = c_1(\eta_i)$. It is then clear that $L(t)$ equals $\prod_1^n (x_i/\tanh x_i)$ not only formally but quite literally. Similarly the expression which is to be proved integral exc 2—forgetting the auxiliary bundle $\eta$ for the the time being—takes the form $e^d \hat{A}(t) = e^d \prod_1^n (\frac{1}{2}x_i/\sinh \frac{1}{2}x_i)$. Hence

$$e^d \hat{A}(t) = \frac{e^d}{2^n} \prod_1^n \frac{\tanh x_i}{\sinh \frac{1}{2}x_i} L(t).$$

Now, it is not hard to see that

$$\frac{\tanh x_i}{\sinh \frac{1}{2}x_i} = \sum^{\infty} b_r (\tanh x_i)^r,$$

where the $b_r$ are integral exc 2. Similarly, if $d' = 2^k d$ is integral, then $e^d = \sum_0^\infty b_r' (\tanh d')^r$, where the $b_r'$ are integral exc 2. Hence, by the integrality of the virtual indexes, $e^d \hat{A}(t)[X]$ is integral exc 2, provided only that $d$ is integral exc 2. (Note also that if in this formula $d$ is replaced $d - c_1(\eta)/2$—which is permissible as $c_1(\eta)$ is integral—one concludes that $e^d T(\eta)[X]$ is integral exc 2, provided $d$ is, and this is essentially the starting point from which the authors proceed in the paper under discussion.) The argument just presented proves the integrality of $e^d \hat{A}(t)$ on an almost complex split manifold but it does not make it clear why the expression $e^d \hat{A}(t)$ should be singled out from all the other integral exc 2 expressions which clearly could have been generated by the procedure. This privileged position of $\hat{A}(t)$ is brought out by the next step of the argument. Let $E$ be the total space of the principal tangent-bundle $t$ of $X$. The structure group of $E$ is $SO(2n)$, $2n = \dim X$. Let $U$ be a maximal torus of $SO(2n)$ and consider the fiber bundle $E/U \xrightarrow{\pi} X$, with fiber $O(2n)/U$. The authors show that $E/U$ can be endowed with a variety of almost complex structures, in all of which it is an almost complex split manifold. Indeed, if $t^\Delta$ is the tangent-bundle of $E/U$, then $t^\Delta = \pi^* t + \xi$, where $\xi$ is the tangent-bundle "along the fibers" of the fibering. By general principles the structure group of $\pi^* t$ can be reduced to the torus $U$, whence $\pi^* t = \varepsilon\eta$, where $\eta \in F_C(E/U)$ splits into a direct sum of $U(1)$-bundles $\eta = \eta_1 + \cdots + \eta_n$. On the other hand the bundle along the fibers is associated to the isotropy representation of $U$ on the tangent space to $G/U$ at the coset $U$. Hence the structure of $\xi$ can also be reduced to $U$, whence, as $U$ is a torus, there exist bundles $\xi_C \in F_C(E/U)$ with $\varepsilon(\xi_C) = \xi$. The authors had already determined all "homogeneous" reductions $\xi_C$ which are possible in the first part of this paper and now apply their findings in this context to establish these facts: (a) There exists a reduction (termed good) $\xi_C$ of $\xi$, for which $T(\xi_C)$ takes the value 1 on the orientation class of $O(2n)/U$. (b) If $w \to w^\natural$ denotes the integration over the fiber in the fiber bundle $E/U \to X$, then $T(\xi_C)^\natural = 1$ whenever $\xi_C$ is a good reduction. It then follows from the properties of the operator $\natural$ that for any $w \in H^{**}(X)$, $w[X] = T(\xi_C)^\natural w[X] = T(\xi_C)\pi^*(w)[E/U]$. Thus one obtains the formula:

$$e^d \hat{A}(t)[X] = T(\xi_C)\pi^*\{e^d \hat{A}(t)\}[E/U].$$

Now the right-hand side is equal to the value of

$$T(\xi_C) e^{\pi^* d} \hat{A}(\pi^* t)$$

on the orientation class of $E/U$. Recalling that $\varepsilon\eta = \pi^* t$, this can be rewritten as $T(\xi_C) \cdot T(\eta) e^{\pi^* d - c/2}$, where $c = c_1(\eta)$. Now from $\varepsilon(\xi_C + \eta) = t^\Delta$—the tangent bundle of $E/U$—and the earlier integrality theorems for split manifolds one concludes that this class is integral exc 2 on $E/U$, and thereby proves the special case of the theorem in mind: If $d$ is integral exc 2, then $e^d \hat{A}(t)$ is integral exc 2 on $[X]$.

The crucial steps of this part of the argument are therefore contained in the assertions (a) and (b). The authors start their paper with a proof of these assertions in a quite general setting. They prove (a) and (b) whenever $E$ is a principal $G$-bundle over $X$, whose structure group $G$ is a compact connected Lie group, and $U$ is a maximal torus of $G$.

With the aid of this general formulation, the divisibility theorem in its final form now follows easily. Indeed let $E$ be the total space of the $U(n)$-bundle $\eta$ over $X$. It has to be shown that $\mathrm{ch}(\eta) e^d \hat{A}(t)$ is integral on $[X]$. For this purpose let $U$ be a maximal torus of $U(n)$, and consider the fibering $E/U \xrightarrow{\pi} X$. Again, if $\xi_C$ is a good unitary reduction of the bundle along the fibers, then $\mathrm{ch}(\eta) e^d \hat{A}(t)[X]$ equals the value of $\mathrm{ch}(\pi^*\eta) e^{\pi^* d} T(\xi_C) \hat{A}(\pi^* t)$ on $E/U$. As the structure group of $\pi^*\eta$ is reducible to $U$, $\mathrm{ch}(\eta) = \sum e^{d_i}$, where $d_i$ are the Chern classes of the components of $\pi^*\eta$. (The $d_i$ are therefore integral.) Thus the class in question takes the form $(\sum e^{d_i}) e^{\pi^* d - c/2} \hat{A}(t')$ where $c = c_1(\xi_C)$ and $t'$ is the tangent bundle of $E/U$. Now the previously proved special case establishes the general case.

Before describing one of the applications which the authors make of the divisibility theorem I will touch very briefly on the connections which the authors derive between the $T$ function and properties of $G/U$, $U$ a maximal torus in $G$, and to which actually the first half of the paper is devoted. For instance, they show that the homogeneous almost complex structure $\xi_C$ on $G/U$ is integrable if and only if $T(\xi_C)[G/U] = 1$. (In general $T(\xi_C) = 1$ or 0.) They also prove the following remarkable fact: If $d \in H^2(G/U; Z)$ then $e^d T(\xi_C)[G/U]$ is either zero or $\pm$ the dimension of an irreducible representation of $G$. The various possibilities are precisely explained and depend on the relation of $d$ with the almost complex structure $\xi_C$. It is at this point that the similarity of the expression $\prod \frac{1}{2}x_i/\sinh \frac{1}{2}x_i$ with the formulae of Hermann Weyl for the character of the irreducible representations of a group comes into play.

To return to the divisibility theorem. By applying the spinor representation to the tangent bundle of $S_{2n}$ the authors construct a $U(2n)$-bundle $\eta$ over $S_{2n}$ which has $n$th Chern class divisible by precisely $(n-1)!$ This upper bound, together with the lower bound given by the divisibility theorem, and a suitable interpretation of the $n$th Chern class over $S_{2n}$, leads the authors to the sequence

$$0 \to Z_{(n-1)!} \to \pi_{2n-2}(U(n-1)) \to \pi_{2n-2}(U(n)) \to 0$$

which they show to be exact mod the class of finite commutative 2-groups. They also obtain analogous statements for the other classical groups.

Let me conclude by summarizing the results of the last section of the paper.

If $K$ is a multiplicative function on $O(n)$-bundles and $X$ is a compact connected oriented manifold, $K(X)$ shall denote the value of $K(t)$ on the orientation class of $X$, $t$ being the tangent bundle. The authors call the function $K$ strictly multiplicative on the class $\mathcal{D}$ if whenever $F \to E \to B$ is a differentiable fibering of compact connected oriented manifolds then $K(\xi)^\natural \in H^0(B)$ where $\xi$ is the tangent bundle along the fibers, and $\natural$ again denotes integration over the fiber. An immediate consequence of this definition is that if $K$ is strictly multiplicative, then $K(E) = K(F) \cdot K(B)$. The authors prove the converse: if for every fibering of the sort described, $K(\text{total space}) = K(\text{base}) \cdot K(\text{fiber})$, then $K$ is strictly multiplicative. From this converse the authors deduce that the function $L$ is strictly multiplicative in $\mathcal{D}$, and finally that, up to a normalization factor, $L$ is the only nontrivial function with this property.

Finally a word about the published work which this paper initiated prior to its own publication. The exact sequence which was described above conflicted, at the time of its discovery, with computations of homotopy theorists and led to a spirited controversy. At present it is known the sequence in question is exact even with regard to the prime 2. Also in the general divisibility theorem the behaviour of the prime 2 is better understood from two quite different points of view. Finally the connection with representations has also been investigated along different lines and the results have in some instances been refined.

*R. Bott* (Cambridge, Mass.)

989:

**Atiyah, M. F.; Hirzebruch, F. Riemann-Roch theorems for differentiable manifolds.** Bull. Amer. Math. Soc. **65** (1959), 276–281.

This is a preliminary account of a theory whose aim is to organize into a coherent framework the consequences of R. Bott's results on the periodicity of the stable homotopy of Lie groups [#987 above].

$F(X)$ being the free abelian group generated by the set of all isomorphism classes $[\xi]$ of unitary vector bundles $\xi$ over the countable finite-dimensional polyhedron $X$, the authors define $K(X)$ to be the quotient group of $F(X)$ by the subgroup generated by the elements of the form $[\xi] - [\xi'] - [\xi'']$, where $\xi = \xi' \oplus \xi''$ (Whitney sum). $K(X)$ and the similarly defined $KO(X)$, using orthogonal vector bundles over $X$, and $KSp(X)$, using symplectic vector bundles over $X$, are the objects of main interest of this theory. Relying on the theory of multiplicative sequences [F. Hirzebruch, *Neue topologische Methoden in algebraischen Geometrie*, Springer, Berlin, 1956; MR **18**, 509] and the relationship between characteristic classes and representation theory of compact Lie groups [A. Borel and F. Hirzebruch, #988 above], the authors sketch the proof of the following theorem.

Let $Y$, $X$ be differentiable closed orientable manifolds with $\dim Y \equiv \dim X \bmod 2$, and let $w_2(Y)$, $w_2(X)$ be their second Stiefel-Whitney classes. Assume a map $f: Y \to X$ is given such that $w_2(Y) - f^* w_2(X) = 0$. Then for every $\eta \in K(Y)$ there exists a $\xi \in K(X)$ such that

$$f_*(\mathrm{ch}(\eta) \cdot \hat{\mathfrak{A}}(Y)) = \mathrm{ch}(\xi) \cdot \hat{\mathfrak{A}}(X);$$

where $f_*$ is the Gysin homomorphism; $\mathrm{ch}(\alpha)$, the Chern character of the bundle $\alpha$, is given by the formal power series expansion of $\sum_k e^{x_k}$ in which the elementary symmetric function $\sigma_q(x_1, \cdots, x_n)$ is to be replaced by the $q$th Chern class $c_q(\alpha)$; and $\hat{\mathfrak{A}}(M)$ is given by the formal power series expansion of $\prod \frac{1}{2}x_k / \sinh \frac{1}{2}x_k$ in which the elementary symmetric function $\sigma_q(x_1^2, \cdots, x_n^2)$ is to be replaced by the $q$th Pontryagin class $p_q$ of the manifold $M$.

Two refinements of this formula are mentioned, one of which reduces to the Riemann-Roch theorem when applied to complex manifolds $Y$ and $X$. Other corollaries are (1) the integrality property of the Todd genus of an almost complex manifold [first proof by J. Milnor, *On the cobordism ring $\Omega^*$, and a complex analogue*; to appear]; (2) invariance properties of the Pontryagin classes of compact oriented differentiable manifolds under homotopy equivalence; (3) a refinement of a theorem of J. Milnor and the reviewer [Proc. Internat. Congress Math. 1958, Cambridge Univ. Press, New York, 1960, 454–458].

*M. A. Kervaire* (New York)

990:

**Milnor, John. Differentiable structures on spheres.** Amer. J. Math. **81** (1959), 962–972.

In a previous paper [Ann. of Math. (2) **64** (1956), 399–405; MR **18**, 498] the author produced differentiable manifolds which are homeomorphic but not diffeomorphic to the 7-sphere $S^7$. Use was made of the fact that $S^3$ is parallelizable. Thus it was not clear a priori that similar examples could be produced in dimensions $\neq 7, 15$.

In this paper a general method of constructing differentiable manifolds which are topological spheres is exhibited. The method consists in matching $D^{m+1} \times S^n$ with $S^m \times D^{n+1}$ by a suitable diffeomorphism $f: S^m \times S^n \to S^m \times S^n$ of the boundaries. ($D^q$ is the $q$-disk in euclidean $q$-space.) An important special case of such diffeomorphisms $f$ is obtained from differentiable maps $f_1: S^m \to \mathrm{SO}_{n+1}$, $f_2: S^n \to \mathrm{SO}_{m+1}$ of spheres into rotation groups by setting $y' = f_1(x) \cdot y$ and $x = f_2(y') \cdot x'$, where $(x', y') = f(x, y)$. The resulting differentiable $(m+n+1)$-manifold is denoted by $M(f_1, f_2)$, and a simple sufficient condition on $f_1, f_2$ is given for $M(f_1, f_2)$ to be homeomorphic to $S^{m+n+1}$.

An invariant $\lambda$ (of the $J$-equivalence class) of $M(f_1, f_2)$ is constructed and turns out to be of interest for dimensions of the form $m = 4r - 1$, $n = 4(k - r) - 1$. (Hence, $m + n + 1 = 4k - 1$.) The value $\lambda(M(f_1, f_2))$ is computed explicitly in terms of the Pontryagin classes of the $\mathrm{SO}_{4(k-r)}$-, resp. $\mathrm{SO}_{4r}$-, bundles over $S^{4r}$, resp. $S^{4(k-r)}$, defined by $f_1$, resp. $f_2$, as characteristic maps. These Pontryagin classes in turn are known by a theorem of R. Bott [Bull. Amer. Math. Soc. **64** (1958), 87–89; MR **21** #1590]. If $M(f_1, f_2)$ is diffeomorphic to $S^{4k-1}$, then $\lambda(M(f_1, f_2)) = 0$. Thus every $(4k-1)$-dimensional $M(f_1, f_2)$ homeomorphic to $S^{4k-1}$ with a $\lambda$ different from zero yields a non-standard differentiable structure on $S^{4k-1}$.

The problem of evaluating the possible values of $\lambda$ (for a given $k$) is shown to amount to solving the following number-theoretic problem. Let $s_k = 2^{2k}(2^{2k-1} - 1)B_k/(2k)!$, where $B_k$ is the $k$th Bernoulli number. Does there exist an integer $r$ satisfying

$$k/3 < r \leq k/2,$$

such that the greatest odd factor $d_{k,r}$ in the denominator of

$$(2r-1)!(2(k-r)-1)! s_r s_{k-r}/s_k$$

is $>1$? If such an $r$ exists, the number $d_{k,r}$ is shown by the author to be a lower bound for the number of distinct differentiable structures on $S^{4k-1}$. The author has checked that the answer to the above question is affirmative for $k=2$ and $4 \leq k \leq 14$. For instance, $d_{5,2}=73$, showing that $S^{19}$ has at least 73 distinct differentiable structures. Similarly, $S^{31}$ is shown to possess at least 16,931,177 distinct differentiable structures.

As usual the author's exposition is excellent.

{Reviewer's note: The above number-theoretic formulation is obtained by supplementing Lemma 5 of the paper under review with the statement on lines 9 and 8 from the bottom of page 348 of the paper by A. Borel and F. Hirzebruch, #988 above.}

*M. A. Kervaire* (New York)

991a:

**Morse, Marston. Differentiable mappings in the Schoenflies problem.** Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 1068–1072.

991b:

**Morse, Marston. Differentiable mappings in the Schoenflies theorem.** Compositio Math. **14**, 83–151 (1959).

B. Mazur [Bull. Amer. Math. Soc. **65** (1959), 59–65] has proved a theorem which implies that a differentiable imbedding $f: S^{n-1} \to E^n$ of a sphere in Euclidean space can be extended to a homeomorphism $F$ of the $n$-disk $D^n$ into $E^n$, where $\partial D^n = S^{n-1}$. The author shows that one can obtain an $F$ which in addition is a diffeomorphism of $D^n$ into $E^n$ except at a single point in the interior of $D^n$. A result of Milnor [Ann. of Math. (2) **64** (1956), 399–405; MR **18**, 498] implies that one cannot for $n=7$ obtain an $F$ which is a diffeomorphism on all of $D^n$. The author's proof is long and technical. In a letter to the reviewer, the author has said that besides new results, he and W. Huebsch have obtained a much shorter proof of the above theorem.

*S. Smale* (Rio de Janeiro)

992:

**Wallace, Andrew H. Sheets of real analytic varieties.** Canad. J. Math. **12** (1960), 51–67.

The author defines a real analytic variety to be a closed subset of Euclidean space $E_n$ which is locally the set of zeros of a finite collection of real analytic functions. A piecewise analytic curve is a union of finitely many analytic arcs joined end to end so that at a common end point $P$ (a "joint") of two arcs $C_1$ and $C_2$, $C_1$ and $C_2$ have distinct tangents at $P$ and $P$ is the end point of no other arc of the curve. The main theorem of the paper is that a piecewise analytic curve (or closed curve) on a real analytic variety $V$ with joints regular points of $V$ can be approximated by an analytic arc (or closed analytic curve) on $V$. To prove the theorem the author projects the piecewise analytic arc onto a certain linear subspace of $E_n$ and applies the corresponding theorem for Euclidean space which is proved in a previous paper. Then by a lifting process he brings the arc back into the variety. A subset $S$ of a real analytic variety is called analytically connected if any two points of $S$ can be joined by an analytic arc in $S$. A sheet of a variety is a maximal analytically connected subset. The author applies his theory to prove several theorems on properties of sheets of a variety.

*S. Smale* (Berkeley, Calif.)

993:

**Bompiani, Enrico. Rappresentazione di elementi differenziali del piano proiettivo.** Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 55–82.

Le présent article s'ajoute aux importantes études que l'auteur a déjà consacrées à la géométrie des éléments différentiels. Il ne comprend d'ailleurs que la première partie du travail, celle consacrée aux éléments différentiels du second ordre dans le plan projectif; une seconde partie, consacrée aux éléments du troisième ordre est annoncée, mais à notre connaissance n'est pas parue. L'auteur rappelle la définition des éléments différentiels ponctuels $E_s$ ou tangentiels $\mathscr{E}_s$ d'ordre $s$ et diverses représentations des $E_1$ du plan; puis il passe à l'introduction de la variété de Gherardelli qui permet de représenter les $E_2$ et $\mathscr{E}_2$ sans exception et il retrouve par des considérations élémentaires, sans recourir à la théorie de la base, quelques caractères algébriques de cette variété. La variété de Gherardelli jouit de remarquables propriétés au point de vue de la théorie des groupes, propriétés auxquelles est consacré le reste de cet article. En particulier l'étude du sous-groupe de stabilité relatif à un point et l'existence d'un groupe $G_9$ qui laisse inchangée la variété sont particulièrement intéressantes.

*M. Decuyper* (Lille)

# AUTHOR INDEX

## PART A


Aczél, János ..... 948
Agnew, Ralph Palmer ..854, 862
Akutowicz, Edwin J. ..... 876
Alardin, Félix ..... 926
Alenicyn, Yu. E. ..... 764
Angell, R. Bradshaw ..... 667
Anger, Gottfried ..... 846
Aniščenko, R. I. ..... 819
Anisimova, E. P. ..... 937
Arya, S. C. ..... 883
Asplund, Edgar ..... 718
Asplund, S. O. ..... 717
Atiyah, M. F. ..... 989
Avishalom, Dov ..... 918
Babič, V. M. ..... 829
Bagemihl, Frederick ..... 672
Bagley, R. W. ..... 747, 964
Bajcsay, P. ..... 790
Bal, Lascu ..... 941
Bandić, Ivan ..... 789
Barsotti, Leo ..... 925
Beckert, Herbert ..... 915
Bellman, Richard ..... 749
Benson, Donald C. ..... 944
Berezanskiĭ, Yu. M. ..... 896
Berlyand, O. S. ..... 868
Berman, D. L. ..... 866
Bessaga, C. ..... 898
Bielecki, Adam ..... 821
Birman, M. S. ..... 817
Bompiani, Enrico ..... 993
Borel, A. ..... 988
Bott, Raoul ..... 987
Bouix, Maurice ..... 827
Bourion, Georges ..... 939
Brauer, Richard ..... 731
Brown, Edgar H., Jr. ..... 979
Bureau, Florent ..... 812
Burnat, M. ..... 814
Butler, John B., Jr. ..... 795
Cairns, S. S. ..... 985
Campanato, Sergio ..... 825
Carlitz, L. .... 689, 690, 694, 788
Charzyński, Zygmunt ..... 849
Chen, Yung-Hoo ..... 886
Cheng, Min-Teh ..... 886
Ciesielski, Z. ..... 863
Cobb, S. M. ..... 676
Cohen, Eckford ..... 691, 692
Colombo, Giuseppe ..... 803
Colombo, Serge ..... 885
Connell, E. H. ..... 976
Conrad, Paul ..... 728, 735
Cooke, R. G. ..... 664
Copeland, Arthur H., Jr ... 979
Coz, Marcel ..... 962
Čudov, L. A. ..... 833
Čžou, Yuĭ-lin' ..... 824
Datzeff, A. B. ..... 794
Davenport, H. ..... 687
Denjoy, Arnaud 752a, 752b, 752c
Derwidué, L. ..... 760
Devinatz, Allen ..... 875
De Vito, Luciano ..... 746
Dijkstra, E. W. ..... 680
Ditkin, V. A. ..... 881, 882
Dixmier, Jacques ..... 741
Doležal, Václav ..... 888
Doob, J. L. ..... 844, 845
Duda, R. ..... 969
Dunham, B. ..... 668
Dyer, E. ..... 986
Džavadov, M. G. ..... 828
Dzyadyk, V. K. ..... 865, 870
Edrei, Albert ..... 770
Eicker, Friedhelm ..... 813
Éĭdel'man, S. D. ..... 831
Erber, Thomas ..... 782
Evgrafov, M. A. ..... 904
Faedo, Sandro ..... 916
Fage, M. K. ..... 834, 835
Faith, Carl ..... 719
Fast, H. ..... 708
Federer, Herbert ..... 961
Feit, Walter ..... 695
Fejes Tóth, L. ..... 934
Filimonov, Yu. M. ..... 804
Fine, N. J. ..... 695
Foiaş, Ciprian ..... 903
Forsythe, G. E. ..... 710
Fridshal, R. ..... 668
Fuchs, Wolfgang H. J. .... 770
Gál, I. S. ..... 966
Gal'perin, I. M. ..... 772
Gary, John ..... 974
Gel'fond, A. O. ..... 702
Gel'man, I. V. ..... 895
Gendler, M. G. ..... 847
Gilbert, Richard C. ..... 902
Gillis, Joseph ..... 785
Gilsdorf, Bernd ..... 963
Godement, Roger ..... 946
Goldberg, Karl ..... 704
Goldhaber, J. K. ..... 673
Gončar, A. A. ..... 867
Gorin, E. A. ..... 967
Gouyon, René ..... 839
Greever, John ..... 977
Griffith, James L. ..... 884
Grünbaum, Adolf ..... 663
Guichardet, M. Alain ..... 910
Günter, N. M. ..... 841
Günther, Paul ..... 820, 950
Habetha, Klaus ..... 843
Haimovici, Adolf ..... 953
Halanay, Aristide ..... 808
Hammer, Preston C. ..... 745
Hampl, Miloslav ..... 686
Hangan, T. ..... 952
Hapaev, M. M. ..... 805
Harary, Frank ..... 675, 678
Haupt, Otto ..... 755
Henrici, P. ..... 710
Herz, J.-C. ..... 713
Hirata, Yukio ..... 897
Hirzebruch, F. ..... 988, 989
Hlodovskiĭ, I. N. ..... 669
Hofmann, Karl Heinrich 742, 743
Holladay, John C. ..... 745
Hong, Imsik ..... 810
Horne, J. G., Jr. ..... 908
Hornfeck, Bernhard ..... 698
Hsu, Chen-Jung ..... 951
Hsu, L. C. ..... 858
Hughes, D. R. ..... 930, 931
Ibragimov, I. I. ..... 871
Iliev, L. ..... 762
Il'in, V. P. ..... 894
Ionescu, D. V. ..... 859
Ionescu-Bujor, Constantin . 920
Ionescu Tulcea, C. ..... 909
Iseki, Shô ..... 700
Ivanov, A. A. ..... 965
Ivanov, V. V. ..... 901
Ivanova, V. M. ..... 965
Jarden, Dov ..... 919
Jarník, Vojtěch ..... 707
Jónsson, Bjarni ..... 685
Jurchescu, Martin ..... 972
Jürimäe, E. ..... 855
Kaasik, Ü. ..... 912, 913
Kaazik, Yu. Ya. = Kaasik, Ü.
Kapilevič, M. B. ..... 815
Karmazina, L. N. ..... 780
Kato, Tosio ..... 711
Khan, Nisar A. ..... 714
Kinoshita, Shin'ichi ..... 984
Kirillov, A. A. ..... 740
Kisyński, Jan ..... 821
Kodama, Yukihiro ..... 978
Kordylewski, J. ..... 850
Koszul, J. L. ..... 982
Kozel, P. T. ..... 732
Král, Josef ..... 756
Krein, M. G. ..... 877
Krylov, V. I. ..... 861
Kuczma, M. ..... 850
Kuiper, N. H. ..... 928
Kurosh, A. G. ..... 727
Kusunoki, Yukio ..... 767
Kuttner, B. ..... 857
Kuzmak, G. E. ..... 807
Landau, Edmund ..... 661
Langer, Rudolph E. ..... 806
Laptev, G. F. ..... 954
Lawton, B. ..... 703
Lederer, G. ..... 744
Lelek, A. ..... 970
Lemleĭn, V. G. ..... 947
Leptin, Horst ..... 730
Lietzmann, W. ..... 922
Littlewood, D. E. ..... 659
Loreh, Lee ..... 864
Lumiste, Ü. ..... 940
Lumiste, Yu. G. = Lumiste, Ü.
Lunc, G. L. ..... 761
Lupanov, O. B. ..... 681
Lyusternik, L. A. ..... 840
MacRobert, T. M. ..... 784
Makai, E. ..... 933
Mąkowski, A. ..... 688
Mamedov, R. G. ..... 899, 900
Mandelbrojt, Szolem ... 750, 751
Manevič, V. A. ..... 923
Mangeron, D. ..... 778
Marcus, Marvin ..... 709, 714
Matlis, Eben ..... 725
Matsushima, Yozô ..... 739
Maurin, K. ..... 811
Mayer-Kalkschmidt, J. ..... 853
McCarthy, P. J. ..... 793
McKnight, J. D. ..... 747, 964
Melzak, Z. A. ..... 889
Menger, Karl ..... 935
Meschkowski, Herbert .... 671
Mickle, E. J. ..... 737
Mikolajska, Z. ..... 801
Miller, J. B. ..... 878
Miller, Maximilian ..... 662
Milmes, Harold Willis ..... 712
Milnor, John ..... 990
Mitropol'skiĭ, Yu. A. ..... 798
Moisil, Gr. C. ..... 809
Monjallon, Albert ..... 660
Morse, Marston ..... 991a, 991b
Mrówka, S. ..... 893
Murakami, Shingo ..... 958
Musielak, J. ..... 863
Naftalevič, A. G. ..... 848
Nagahara, Takasi ..... 721
Nagata, Masayoshi ..... 720
Nakamura, Yoshio ..... 701
Nelson, Edward ..... 907
Nemyckiĭ, V. V. ..... 802
Neumann, M. ..... 932
Nirenberg, L. ..... 823
Obreškov, N. ..... 706, 851
Ogata, Hayao ..... 897
Ögmundsson, Vilhjálmur .. 722
Olech, C. ..... 797
Olver, F. W. J. ..... 666, 781
Onodera, Takesi ..... 721
Opial, Z. ..... 799
Osborn, Howard ..... 914
Osserman, Robert ..... 766
Parker, E. T. ..... 674, 729
Pauc, Christian Y. ..... 755
Pelczyński, A. ..... 898
Pickert, Günter ..... 929
Pogorzelski, W. ..... 763, 832
Pommerenke, Chr. ..... 758, 759
Povolockiĭ, A. I. ..... 757
Pressman, A. Ya. ..... 868
Prudnikov, A. P. ..... 881
Pták, Vlastimil ..... 890, 891
Rabinovič, Yu. L. ..... 805
Radlow, James ..... 880
Radó, T. ..... 737
Redheffer, Raymond ..... 792
Reghiş, M. ..... 800
Reiche, Erhard ..... 938
Reid, William T. ..... 791
Reinhart, Bruce L. ..... 826
Reissig, Rolf ..... 796
Remizova, M. P. ..... 771
Rényi, A. ..... 773, 774
Restle, Frank ..... 968
Ritchie, R. W. ..... 724
Rivlin, R. S. ..... 715
Robin, Louis ..... 779
Rolewicz, S. ..... 898
Rose, Alan ..... 670
Ross, Ian C. ..... 675
Rothe, E. H. ..... 911
Roy, Bernard ..... 677
Ryžkov, V. V. ..... 945
Sadowska, D. ..... 887
Saltykow, N. ..... 838
Šaškin, Yu. A. ..... 842
Saxer, Walter ..... 665
Schafer, R. D. ..... 723
Schoenfeld, Lowell ..... 869
Schweizer, Berthold ..... 935
Sears, D. B. ..... 818
Seeley, R. T. ..... 905
Seidman, Thomas I. ..... 892
Sen, D. K. ..... 957
Sierpiński, W. ..... 696
Sikorski, R. ..... 684
Simon, Arthur B. ..... 909
Sklar, Abe ..... 935
Skof, Fulvia ..... 924
Skvorcov, V. A. ..... 753
Ślebodziński, W. ..... 917
Slobodeckiĭ, L. N. ..... 830
Smart, D. R. ..... 856
Sobolev, S. L. ..... 836
Sonner, Johann ..... 726
Sorokina, N. G. ..... 874
Spencer, A. J. M. ..... 715
Sretenskiĭ, L. N. ..... 783
Staaheff, James ..... 980
Stein, Erwin Walter ..... 921
Stelleckiĭ, I. V. ..... 682, 683
Sternberg, Shlomo ..... 738, 826
Stinespring, W. Forrest ... 907
Stoka, Marius I. ..... 955
Stroud, A. H. ..... 860
Su, Buchin ..... 936
Sunouchi, Gen-ichiro ..... 872
Suvorov, G. D. ..... 975
Suzuki, Michio ..... 731
Świerczkowski, S. ..... 708, 927
Sz.-Nagy, Béla ..... 903
Tabata, Fujio ..... 949
Takahashi, Shuichi ..... 733
Tamme, E. E. ..... 913
Tanaka, Tadashi ..... 973
Terehov, A. A. ..... 734
Tersenov, S. A. ..... 822
Thomas, Emery ..... 981
Titus, Charles J. ..... 971
Tominaga, Hisao ..... 721
Touchard, Jacques ..... 693
Trušin, G. I. ..... 942
Tumarkin, G. C. ..... 777
Tutte, W. T. ..... 679
Ugodčikov, A. G. ..... 765
Urbanik, K. ..... 748
Ursell, F. ..... 666
Valiron, Georges ..... 769
Varga, O. ..... 716
Varshney, O. P. ..... 873
Vinberg, È. B. ..... 956
Vinogradov, I. M. ..... 699
Vinokurov, V. G. ..... 754
Višik, M. I. ..... 840

Vitušķin, A. G. ..... 775
Wakakuwa, Hidekiyo ..... 959
Wallace, Andrew H. ..... 992
Walsh, J. L. ..... 776
Watari, Chinami ..... 872
Watson, G. L. ..... 705
Weisner, Louis ..... 786, 787
Weiss, George ..... 785
Wells, J. H. ..... 852
Wermer, John ..... 768
Willmore, Thomas James .. 960
Wu, Wen-tsün ..... 983
Yurimyaė, È. I. = Jūrimāe, E.
Zajtz, A. ..... 943
Zareckiĭ, K. A. ..... 736
Ždanov, G. A. ..... 837
Zeckendorf, E. ..... 697
Želobenko, D. P. ..... 906
Zemanian, Armen H. ..... 879
Žislin, G. M. ..... 816